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Review article

Genetic coding and united-hypercomplex systems in the models of algebraic biology

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ABSTRACT

Structured alphabets of DNA and RNA in their matrix form of representations are connected with Walsh functions and a new type of systems of multidimensional numbers. This type generalizes systems of complex numbers and hypercomplex numbers, which serve as the basis of mathematical natural sciences and many technologies. The new systems of multi-dimensional numbers have interesting mathematical properties and are called in a general case as "systems of united-hypercomplex numbers" (or briefly "U-hypercomplex numbers"). They can be widely used in models of multi-parametrical systems in the field of algebraic biology, artificial life, devices of biological inspired artificial intelligence, etc. In particular, an application of U-hypercomplex numbers reveals hidden properties of genetic alphabets under cyclic permutations in their doublets and triplets. A special attention is devoted to the author's hypothesis about a multi-linguistic in DNA-sequences in a relation with an ensemble of U-numerical sub-alphabets. Genetic multi-linguistic is considered as an important factor to provide noise-immunity properties of the multi-channel genetic coding. Our results attest to the conformity of the algebraic properties of the U-numerical systems with phenomenological properties of the DNA-alphabets and with the complementary device of the double DNA-helix. It seems that in the modeling field of algebraic biology the genetic-informational organization of living bodies can be considered as a set of united-hypercomplex numbers in some association with the famous slogan of Pythagoras "the numbers rule the world".

Contents

1. Introduction

The main task of the mathematical natural sciences is the creation of mathematical models of natural systems. Development of models and formalized theories depends highly on those mathematical notions and instruments, on which they are based.

In the beginning of the XIX century the following opinion existed: the world possesses the single real geometry (Euclidean geometry) and the single arithmetic. But this opinion was neglected after the discovery of non-Euclidean geometries and of quaternions by Hamilton. Science understood that different natural systems could possess their own individual geometries and their own individual arithmetic (see this theme in the book (Kline, 1980)). The

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$$Z = a_0 \bullet 1 + a_1 \bullet i, \quad \text{or } Z = a_0 \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_1 \bullet \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \\ -a_1 & a_0 \end{bmatrix}$$

•	1	i
1	1	i
i	i	-1

Fig 1. Left: representations of complex numbers Z in its linear and matrix forms where the matrix [1,0; 0, 1] = E is the identity matrix and the matrix [0,1; −1, 0] represents the imaginary unit “i”. Right: the multiplication table of basic elements of complex numbers.

example of Hamilton, who was unlucky during 10 years in his attempts to solve the task of description of transformations of 3D space by means of 3-dimensional algebras, is a very demonstrative one. This example says that, if a scientist does not guess correctly what types of numeric systems are adequate for the natural system investigated by him, he can waste many years without any result. One can add that effectiveness of analytical models of geometrical and physical-geometrical properties of separate natural systems (including laws of conservation, theories of oscillations and waves, theories of potentials and fields, etc.) depends on adequatecy of used numeric systems.

The idea about special mathematical peculiarities of living matter exists long ago. For example Vernadsky (1965) put forward the hypothesis on a non-Euclidean geometry of living nature but without any concrete definition of the type of such geometry. We also believe that living bodies live in their own bio-information space, which has specific algebraic and geometric properties. In our opinion many difficulties of modern science to understand genetic and other biological systems are determined by approaches to them from the non-adequate algebraic standpoints, which were developed formerly for other systems.

But how one can find appropriate algebraic approaches in biology if various species of organisms differ from each other so significantly in their morphogenetic and many other features? The discovery of the genetic code, basic features of which are general for all biological organisms, has allowed hoping that such mathematical problem can be solved by means of investigation of genetic code structures. It seems an important task to investigate from different standpoints what systems of multi-dimensional numbers are connected or can be connected with ensembles of parameters of the genetic code and with relevant bio-information spaces. Some results of such investigation are presented in this article.

Various kinds of complex and hypercomplex numbers – complex numbers, double (or hyperbolic) numbers, quaternions and others – are used in different branches of modern science. They have played the role of the magic tool for development of theories and calculations in problems of heat, light, sounds, fluctuations, elasticity, gravitation, magnetism, electricity, current of liquids, quantum-mechanical phenomena, special theory of relativity, nuclear physics, etc. For example, in physics thousands of works – only in XX century – were devoted to quaternions of Hamilton (their bibliography is in (Gsponer and Hurni, 2008)).

Each kind of hypercomplex numbers has its own multiplication table of basic elements. As known, complex and hypercomplex numbers can be written not only in linear forms but also in matrix forms of their representation (in cases of associative algebras).

Hypercomplex numbers are a generalization of complex numbers. The latter have one real unit “1” and one imaginary unit “i” with the property $i^2 = -1$. Complex numbers are written in a linear form and also in a matrix form (Fig. 1).

As known, hypercomplex numbers have a single real unit and additional quantity of imaginary units, all of which in the total are called basic elements of hypercomplex numbers. The expression (1) shows their linear form of writing in general case:

$$a_0E + a_1e_1 + a_2e_2 + \ldots\ldots + a_ne_n \tag{1}$$

where E denotes the real unit (or the identity matrix E in the case of the matrix representation of hypercomplex numbers), $e_1, e_2, \ldots, e_n$ denote imaginary units, $a_0, a_1, \ldots, a_n$ denote real coefficients. The set of these basic elements ($E, e_1, e_2, \ldots, e_n$) should be closed under multiplication. This means that multiplication of any two basic elements with each other always gives again one of these elements or their linear superposition (Kantor and Solodovnikov, 1989). The results of the multiplication of the basic elements are represented in the corresponding multiplication table, which is individual for each type of hypercomplex numbers. If a system (1) does not possess the real unit E, which is mutual for all terms in the expression (1), it is not a system of hypercomplex numbers.

In the result of our study of systems of structured alphabets of DNA and RNA in matrix forms of their representations (Petoukhov, 2008, 2012, 2014, 2016a,b, 2017; Petoukhov and He, 2010; Petoukhov and Petukhova, 2017a,b), we have revealed that these alphabetic structures are connected with special ($2^n \times 2^n$)-matrix operators. Various decompositions of these operators show their connection with special 2^n-dimensional numeric systems with the following features:

1) they are not hypercomplex numbers in the whole because of absence of the identity matrix E in their decompositions (that is the general real unit is absent among basic elements of such systems);
2) they contain two or more blocks, each of whose is a sparse matrix and represents complex numbers or hypercomplex numbers. Each of blocks contains an individual set of basic elements, which play role of its own real unit and imaginary units but only inside this separate block; by this reason we call them as local-real units and local-imaginary units.

In other words, two or more different systems of complex or hypercomplex numbers exist in this case not separately but are combined together into a single structure by means of a summation of sparse matrices. The author calls such multi-dimensional numeric systems as “multi-block united-hypercomplex systems” (or more briefly, “united-hypercompex numbers” or simply “U-numbers”). Such systems of united-hypercomplex numbers are

$$H = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad H^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad H^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Fig. 2. The beginning of the tensor (or Kronecker) family of Hadamard matrices $H^{(n)}$, where (n) means a tensor power. Black cells correspond to components “+1”, white cells – to “-1”.

[C A; T G] =

C	A
T	G

;

[C A; T G]$^{(2)}$ =

CC	CA	AC	AA
CT	CG	AT	AG
TC	TA	GC	GA
TT	TG	GT	GG

[C A; T G]^3 =

CCC Pro	CCA Pro	CAC His	CAA Gln	ACC Thr	ACA Thr	AAC Asn	AAA Lys
CCT Pro	CCG Pro	CAT His	CAG Gln	ACT Thr	ACG Thr	AAT Asn	AAG Lys
CTC Leu	CTA Leu	CGC Arg	CGA Arg	ATC Ile	ATA Met	AGC Ser	AGA Stop
CTT Leu	CTG Leu	CGT Arg	CGG Arg	ATT Ile	ATG Met	AGT Ser	AGG Stop
TCC Ser	TCA Ser	TAC Tyr	TAA Stop	GCC Ala	GCA Ala	GAC Asp	GAA Glu
TCT Ser	TCG Ser	TAT Tyr	TAG Stop	GCT Ala	GCG Ala	GAT Asp	GAG Glu
TTC Phe	TTA Leu	TGC Cys	TGA Trp	GTC Val	GTA Val	GGC Gly	GGA Gly
TTT Phe	TTG Leu	TGT Cys	TGG Trp	GTT Val	GTG Val	GGT Gly	GGG Gly

Fig. 3. The tensor family of genetic matrices [C, A; T, G]$^{(n)}$ (n = 1,2,3). Black cells of the matrices contain 32 triplets with strong roots and also 8 doublets, which play the role of such strong roots. White cells of the matrices contain 32 triplets with weak roots and also 8 doublets, which play the role of such weak roots (see explanation in the text). Encoded amino acids and stop-signals of protein synthesis are shown for the case of the Vertebrate mitochondrial genetic code that is the most symmetrical among known variants of the genetic code (initial data are taken from http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi).

new mathematical instruments, which possess interesting properties to model biological phenomena and multi-parametrical systems in general. One can think that united-hypercomplex numbers play an important role in bio-information phenomena since they are connected with the phenomenology of alphabets of DNA and RNA. Some examples of application of united-hypercomplex numeric systems to model biological phenomena are given. From the modeling standpoint, the hypothesis arises that inherited bio-informational aspects of organization of living matter is ruled by systems of 2^n-block U-hypercomplex numbers. The described results evoke associations with the famous Pythagoras's slogan “numbers rule the world”.

2. Genetic alphabets and Walsh-representations of genetic matrices

Genetic information, which is written in DNA and RNA at the quantum-mechanical level, dictates properties of whole organisms at very other levels in conditions of strong interferences and noises. This dictat is realized by means of unknown algorithms of multi-channel noise-immunity coding. For example, in accordance with Mendel‘s laws of independent inheritance of traits, colors of human skin, eye and hairs are genetically defined independently. So, each living organism can be hypothetically considered an algorithmic machine of multi-channel noise-immune coding. The similar problem of multi-channel noise-immune coding is solved in engineering by means of Hadamard matrices, rows of which are Walsh functions. Relevant mathematical solutions are used for example to transmit photos of the Martian surface to Earth by means of electromagnetic signals traveling through millions kilometers of interference. Below we apply some formalisms of this mathematics to study genetic structures. We intensively use tensor multiplication of matrices (see Appendix A) taking into account additionally that DNA-molecules belong to the level of quantum mechanics, where the tensor multiplication is the important operation: when considering a quantum system consisting of two subsystems, the whole space of states is constructed by the tensor product of their states.

$$W_2 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}; \quad W_3 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 4. Walsh-representations W_2 and W_3 of the mosaic matrices [C, A; T, G]$^{(2)}$ and [C, A; T, G]$^{(3)}$ from Fig. 3, where each row is one of Walsh functions.

In noise-immune coding and also in quantum mechanics, Hadamard matrices play significant roles. By definition, a Hadamard matrix is a square matrix H with entries ±1, which satisfies $H \bullet H^T = n \bullet E$, where H^T – transposed matrix, E – identity matrix. Tensor (or Kronecker) exponentiation of Hadamard (2×2)-matrix H generates a tensor family of Hadamard $(2^n \times 2^n)$-matrices $H^{(n)}$ (Fig. 2), rows and columns of which are Walsh functions (Ahmed and Rao, 1975).

All living organisms are identical from the point of view of the molecular foundations of genetic coding of sequences of amino acids in proteins. This coding is based on molecules of DNA and RNA. In DNA the genetic information is recorded using different sequences of 4 nitrogenous bases, which play the role of letters of the alphabet: adenine A, guanine G, cytosine C and thymine T (uracil U is used in RNA instead of thymine T). The list of DNA-alphabets includes also alphabets of 16 doublets and 64 triplets (generally speaking, each DNA-alphabet of n-plets contains 4^n n-plets).

Hadamard matrices in Fig. 2 also consist of 4, 16 and 64 entries. In a general case a Hadamard $(2^n \times 2^n)$-matrix consists 4^n entries. By analogy we represent the system of DNA-alphabets in a form of the tensor family of square genetic matrices [C, A; T, G]$^{(n)}$ in Fig. 3.

Black and white cells of genetic matrices [C, A; T, G]$^{(2)}$ and [C, A; T, G]$^{(3)}$ in Fig. 3 reflect the known phenomenon of segregation of the set of 64 triplets into two equal sub-sets on the basis of strong and weak roots, i.e., the first two positions in triplets (Rumer, 1968): a) black cells contain 32 triplets with strong roots, i.e., with 8 “strong” doublets AC, CC, CG, CT, GC, GG, GT, TC; b) white cells contain 32 triplets with weak roots, i.e., with 8 “weak” doublets AA, AG, AT, CA, GA, TA, TG, TT. Code meanings of triplets with strong roots do not depend on the letters on their third position; code meanings of triplets with weak roots depend on their third letter (see details in (Petoukhov, 2016a)). Each of matrices [C, A; T, G]$^{(2)}$ and [C, A; T, G]$^{(3)}$ has a cross-shaped mosaic structure, where both quadrants along each of diagonals are identical from the standpoint of their mosaic.

Fig. 3 shows an unexpected phenomenon of a symmetrical disposition of black and white triplets in the genetic matrix [C, T; A, G]$^{(3)}$, which was constructed formally without any mention about strong and weak roots, amino acids and the degeneracy of the genetic code:

1) the left and right halves of the matrix mosaic are mirror-anti-symmetric each to other in its colors: any pair of cells, disposed by mirror-symmetrical manner in the halves, possesses the opposite colors;
2) both quadrants along each diagonals are identical from the standpoint of their mosaic and correspondingly the matrix has a cross-shaped structure;
3) the mosaics of all rows have meander configurations (each row has black and white fragments of equal lengths) and they are identical to mosaics of some Walsh functions (see mosaics of rows in Fig. 2), which coincide with Rademacher functions as the particular cases of Walsh functions (Petoukhov, 2008; Petoukhov and He, 2010);

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&0&0\\0&-1&0&0\end{bmatrix}+\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&1&0\\0&0&0&-1\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}=e_0+e_1+e_2+e_3.$$

•	e_0	e_1
e_0	e_0	e_1
e_1	e_1	$-e_0$

•	e_2	e_3
e_2	e_2	e_3
e_3	e_3	$-e_2$

Fig. 5. Left: the decomposition of the matrix W_2 (Fig. 4) into 4 sparse matrices e_0, e_1, e_2, e_3. Right: multiplication tables for the pair e_0 and e_1 and for the pair e_2 and e_3 coincide with the multiplication table of basic elements of complex numbers.

$$Z_L=\begin{bmatrix}a_0&-a_1&0&0\\a_1&a_0&0&0\\a_0&-a_1&0&0\\-a_1&-a_0&0&0\end{bmatrix};\qquad Z_L^{-1}=((a_0^2+a_1^2)^{-1})\bullet(a_0e_0-a_1e_1)=((a_0^2+a_1^2)^{-1})\bullet\begin{bmatrix}a0,&a1,&0,&0\\-a1,&a0,&0,&0\\a0,&a1,&0,&0\\a1,&-a0,&0,&0\end{bmatrix}$$

$$Z_R=\begin{bmatrix}0&0&a_2&-a_3\\0&0&-a_3&-a_2\\0&0&a_2&-a_3\\0&0&a_3&a_2\end{bmatrix};\qquad Z_R^{-1}=((a_2^2+a_3^2)^{-1})\bullet(a_2e_2-a_3e_3)=((a_2^2+a_3^2)^{-1})\bullet\begin{bmatrix}0,&0,&a_2,&a_3\\0,&0,&a_3,&-a_2\\0,&0,&a_2,&a_3\\0,&0,&-a_3,&a_2\end{bmatrix}$$

Fig. 6. Left: two systems of complex numbers $Z_L = a_0e_0 + a_1e_1$ and $Z_R = a_2e_2 + a_3e_3$ in their form of (4×4)-matrices. Right: inverse matrices Z_L^{-1} and Z_R^{-1} in relation to matrices e_0 and e_2, which play the role of identity matrices in the systems Z_L and Z_R correspondingly.

$$Z_LZ_R=[0,\ 0,\ a_0a_2+a_1a_3,\ a_1a_2-a_0a_3;\ 0,\ 0,\ a_1a_2-a_0a_3,\ -a_0a_2-a_1a_3;\ 0,\ 0,\ a_0a_2+a_1a_3,\ a1a_2-a_0a_3;\ 0,\ 0,\ a_0a_3-a_1a_2,\ a_0a_2+a_1a_3]$$

$$Z_RZ_L=[a_0a_2+a_1a_3,\ a_0a_3-a_1a_2,\ 0,\ 0;\ a_1a_2-a_0a_3,\ a_0a_2+a_1a_3,\ 0,\ 0;\ a_0a_2+a_1a_3,\ a_0a_3-a_1a_2,\ 0,\ 0;\ a_0a_3-a_1a_2,\ -a_0a_2-a_1a_3,\ 0,\ 0]$$

Fig. 7. Multiplication of two matrices, where one matrix is taken from the system Z_L and the second matrix is taken from Z_R (Fig. 6), is non-commutative.

$$Z_2=Z_L+Z_R=[a_0,\ -a_1,\ a_2,\ -a_3;\ a_1,\ a_0,\ -a_3,\ -a_2;\ a_0,\ -a_1,\ a_2,\ -a_3;\ -a_1,\ -a_0,\ a_3,\ a_2];$$

•	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	$-e_3$	e_2
e_2	e_0	e_1	e_2	e_3
e_3	$-e_1$	e_0	e_3	$-e_2$

Fig. 8. Left: the matrix representation of united-complex numbers $Z_2 = Z_L + Z_R$. Right: the multiplication table of 4 sparse matrices e_0, e_1, e_2, e_3 from Fig. 5.

4) each pair of adjacent rows with decimal numeration 0–1, 2–3, 4–5, 6–7 has an identical mosaic (the realization of the principle "even-odd").

It should be noted that a huge quantity $64! \approx 10^{89}$ of variants exists for locations of 64 triplets in a separate (8×8)-matrix. For comparison, the modern physics estimates time of existence of the Universe in 10^{17} s. It is obvious that an accidental disposition of black and white triplets (and corresponding amino acids) in a (8×8)-matrix will almost never give symmetries. But in our approach this matrix of 64 triplets (Fig. 3) is not a separate matrix, but it is one of members of the tensor family of matrices of genetic alphabets, and in this case wonderful symmetries are revealed in the location of black and white triplets. These symmetries testify that the location of black and white triplets in the set of 64 triplets is not accidental. By author's opinion these symmetries can be explained by connections of genetic alphabets with systems of multi-dimensional numbers desribed below.

Walsh functions contain only components "+1" and "−1" (Ahmed and Rao, 1975). Since mosaics of rows of genetic matrices [C, A; T, G]$^{(2)}$ and [C, A; T, G]$^{(3)}$ in Fig. 3 coincide with mosaics of Walsh functions, one can numerically represent these symbolic matrices in a form of numeric matrices, which contain "+1" in their black cells and "-1" in their white cells (so called the "Walsh-representation"). Fig. 4 shows such Walsh-representations W_2 and W_3 of the matrices [C, A; T, G]$^{(2)}$ and [C, A; T, G]$^{(3)}$ from Fig. 3.

Below we will show that the Walsh-representations of these genetic matrices possess rich algebraic properties and are connected with a very special type of multi-dimensional numeric systems and matrix operators, which can be used to model genetic phenomena.

3. United-complex numbers and the genetic matrix of 16 doublets

Let us consider the Walsh-representation W_2 (Fig. 4) of the matrix of 16 doublets [C, A; T, G]$^{(2)}$. The matrix W_2 can be decomposed into sum of 4 sparse matrices e_0, e_1, e_2, e_3, for example, in the way shown on Fig. 5. It is interesting that this variant of the decomposition consists of two sets of matrices, each of which is closed under multiplication unexpectedly: the first set consists of matrices e_0 and e_1; the second set – matrices e_2 and e_3. Fig. 5 shows multiplication tables for both sets, in each of which multiplication between its elements always gives an element from the same set. These multiplication tables coincide with the multiplication table of systems of complex numbers in Fig. 1.

It means that each of expressions $Z_L = a_0e_0 + a_1e_1$ and $Z_R = a_2e_2 + a_3e_3$ represents its own system of complex numbers in the unusual form of the sparse (4×4)-matrix with their 2 independent parameters a_0, a_1 and a_2, a_3 correspondingly (Fig. 6). It can be formulated also in the following way: each of the systems Z_L and Z_R is isomorphic to the classical system of complex numbers. Inside the system Z_L, the matrix $e_0 + e_1$ represents the complex number with unit coordinates ($a_0 = a_1 = 1$). Inside the system Z_R, the matrix $e_2 + e_3$ represents the complex number with unit coordinates ($a_2 = a_3 = 1$).

$$\begin{bmatrix} a_0, -a_1, & a_2, -a_3 \\ a_1, & a_0, -a_3, -a_2 \\ a_0, -a_1, & a_2, -a_3 \\ -a_1, -a_0, & a_3, & a_2 \end{bmatrix} \bullet \begin{bmatrix} b_0, -b_1, & b_2, -b_3 \\ b_1, & b_0, -b_3, -b_2 \\ b_0, -b_1, & b_2, -b_3 \\ -b_1, -b_0, & b_3, & b_2 \end{bmatrix} = \begin{bmatrix} c_0, -c_1, & c_2, -c_3 \\ c_1, & c_0, -c_3, -c_2 \\ c_0, -c_1, & c_2, -c_3 \\ -c_1, -c_0, & c_3, & c_2 \end{bmatrix}$$

where $c_0 = a_0b_0 - a_1b_1 + a_2b_0 + a_3b_1$,
$c_1 = a_0b_1 + a_1b_0 + a_2b_1 - a_3b_0$,
$c_2 = a_0b_2 + a_1b_3 + a_2b_2 - a_3b_3$,
$c_3 = a_0b_3 - a_1b_2 + a_2b_3 + a_3b_2$

Fig. 9. Multiplication of two matrices with the described structure of 4-parametrical U-complex numbers $Z_2 = Z_L + Z_R$ (from Fig. 7) gives a new matrix of the same location of its 4 parameters c_0, c_1, c_2 and c_3.

$[x_0,x_1,x_2,x_3] \bullet Z_L = [a_0x_0+a_0x_2+a_1x_1-a_1x_3,$	$a_0x_1-a_1x_0-a_0x_3-a_1x_2, \quad 0, \quad 0]$
$[x_0,x_1,x_2,x_3] \bullet Z_R = [0, \quad 0,$	$a_2x_0+a_2x_2-a_3x_1+a_3x_3, \quad a_2x_3-a_3x_0-a_2x_1-a_3x_2]$

Fig. 10. Multiplication of matrix operators Z_L and Z_R (Fig. 7) with a vector $X = [x_0,x_1,x_2,x_3]$ from the right side leads to a selective manage of subspaces.

$Z_L \bullet [x_0,x_1,x_2,x_3]^T =$	$[a_0x_0-a_1x_1; \quad a_0x_1+a_1x_0; \quad a_0x_0-a_1x_1; \quad -(a_0x_1+a_1x_0)]$
$Z_R \bullet [x_0,x_1,x_2,x_3]^T =$	$[a_2x_2-a_3x_3; \quad -(a_2x_3+a_3x_2); \quad a_2x_2-a_3x_3; \quad a_2x_3+a_3x_2]$

Fig. 11. Multiplication of the matrix operators Z_L and Z_R (Fig. 7) with a vector $X = [x_0,x_1,x_2,x_3]$ from the left side leads to effects of twinning.

The classical identity matrix E = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1] is absent in the set of matrices Z_L and Z_R, where each matrix has zero determinant. Consequently the usual notion of the inverse matrix Z_L^{-1} or Z_R^{-1} in relation to E (as $Z_LZ_L^{-1} = E$ or $Z_RZ_R^{-1} = E$) can't be defined in accordance with the famous theorem about inverse matrices for matrices with zero determinant in the case of the complete set of matrices (Bellman, 1960, Chapter 6, § 4). In our case we analyze not the complete set of (4 × 4)-matrices but very limited special sets of matrices Z_L and Z_R. The set Z_L has the matrix e_0 (Fig. 5), which possesses all properties of the identity matrix for any matrix Z_L since $e_0Z_L = Z_Le_0 = Z_L$ and $e_0^2 = e_0$. In the frame of the set of matrices Z_L, where locally the matrix e_0 plays the role of the real unit, one can define – for any non-zero matrix Z_L – its inverse matrix Z_L^{-1} in relation to the matrix e_0 on the basis of equations: $Z_LZ_L^{-1} = Z_L^{-1}Z_L = e_0$. (Fig. 6, right).

By analogy, the set of matrices Z_R has the matrix e_2 (Fig. 5), which possesses all properties of the identity matrix for any matrix Z_R since $e_2Z_R = Z_Re_2 = Z_R$ and $e_2^2 = e_2$. In the frame of the set of matrices Z_R, where locally the matrix e_2 plays the role of the real unit, one can define – for any non-zero matrix Z_R – its inverse matrix Z_R^{-1} in relation to the matrix e_2 on the base of equations: $Z_RZ_R^{-1} = Z_R^{-1}Z_R = e_2$ (Fig. 6, right).

Multiplication of two members from the same set Z_L (or Z_R) is commutative as it is true for complex numbers. But multiplication of two members from different sets (one member from Z_L and one member from Z_R) is not commutative and gives a new matrix from one of these sets (Fig. 7).

Multiplication of the complex number $Z_L = a_0e_0 + a_1e_1$ with its conjugate complex number a_0e_0-a_1e_1 gives the norm of this complex number $(a_0^2 + a_1^2)e_0$. Multiplication of the complex number $Z_R = a_2e_2 + a_3e_3$ with its conjugate complex number a_2e_2-a_3e_3 gives the norm of this complex number: $(a_2^2 + a_3^2)e_2$.

One should emphasize that the total set of sparse matrices e_0, e_1, e_2, e_3 does not contain the identity matrix E = [1,0,0,0; 0,1,0,0; 0,0,1,0; 0,0,0,1] and therefore the total sum $Z_2 = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ does not represent a system of hypercomplex numbers. It contains two blocks of complex numbers $(a_0e_0 + a_1e_1)$ and $(a_2e_2 + a_3e_3)$. By this reason it is one of many types of 2^n-block united-complex and united-hypercomplex numbers, which – as we believe – reflect the specificity of inherited structures of living organisms and which – as we propose – should be used to model biological phenomena in the field of algebraic biology.

The set of matrices e_0, e_1, e_2, e_3 is closed under multiplication and defines the multiplication table of U-complex numbers Z_2 in Fig. 8. In this system Z_2, the matrices e_0 and e_1 are conditionally called as the local-real unit and the local-imaginary unit correspondingly in the block $Z_L = a_0e_0 + a_1e_1$; the matrices e_2 and e_3 are called as the local-real unit and the local-imaginary unit in the block $Z_R = a_2e_2 + a_3e_3$.

This multiplication table has a cross-shaped struture with an element of a fractal plexus in relative locations of sets of the basic units:

- the structure of both (2 × 2)-quadrants along the main diagonal coincides with the multiplication table of basic elements of complex numbers; in opposite to this, the structure of both (2 × 2)-quadrants along the second diagonal coincides with the known matrix representation of complex numbers [a,b; −b,a] (Fig. 1);
- all cells of the main diagonal of the table contain only local-real units e_0 and e_2; all cells of the second diagonal contain only local-imaginary units e_1 and e_3;
- each of the four (2 × 2)-sub-quadrants of the table also has a cross-shaped sructure since its main diagonal contains only local-real units e_0 or e_2 and its second diagonal contains only local-imaginary units e_1 and e_3.

Ordinary united-hypercomplex systems can be written in the following linear form (2) in the case of using ordinary multiplication in definition of their separate blocks (below we will also meet tensor-U-hypercomplex numbers, where tensor multiplication participates in definition of separate blocks of united-hypercomplex numbers):

$$(a_0E_1 + a_1e_1 + \ldots + a_ke_k) + (b_0E_2 + b_1q_1 + \ldots + b_mq_m) + \ldots + (d_0E_n + d_1j_1 + \ldots + d_pj_p) \quad (2)$$

where each of expressions in brackets represents its own complex or hypercomplex system; $E_1, E_2, \ldots, E_n$ are local-real units; $e_1, \ldots, e_k$, $q_1, \ldots, q_m$, $j_1, \ldots, j_p$ are local-imaginary units; $a_0 \cdot a_k$, $b_0, \ldots, b_m$, $d_0, \ldots, d_p$ – real coefficients. It is obvious that U-hypercomplex systems (2) are a generalization of hypercomplex systems (1). The known Frobenius theorem for hypercomplex systems (Kantor and Solodovnikov, 1989) does not correspond to U-hypercomplex systems in the whole.

The system of U-hypercomplex numbers $Z_2 = Z_L + Z_R$ has operations of addition, subtraction and non-commutative multiplication (Fig. 9): applications of any of these operations for 2-block U-complex numbers Z_2 gives a new 2-block U-complex number of the same type. But this system has no operation of division.

$$[x_0,x_1,x_2,x_3]\bullet Z_2 = [a_0x_0+a_0x_2+a_1x_1-a_1x_3,\quad a_0x_1-a_1x_0-a_0x_3-a_1x_2,\quad a_2x_0+a_2x_2-a_3x_1+a_3x_3,\quad a_2x_3-a_3x_0-a_2x_1-a_3x_2]$$

$$Z_2\bullet[x_0,x_1,x_2,x_3]^T = [a_0x_0-a_1x_1+a_2x_2-a_3x_3;\quad a_0x_1+a_1x_0-a_2x_3-a_3x_2;\quad a_0x_0-a_1x_1+a_2x_2-a_3x_3;\quad a_2x_3-a_1x_0-a_0x_1+a_3x_2]$$

Fig. 12. Multiplication of the matrix operators $Z_2 = Z_L + Z_R$ with a vector $X = [x_0,x_1,x_2,x_3]$ from the right side and from the left side.

$$\begin{bmatrix}1&0&0&-1\\0&1&-1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+\begin{bmatrix}0&-1&1&0\\1&0&0&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&-1&1&0\\-1&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&-1\\0&-1&1&0\end{bmatrix}=q_0+q_1+q_2+q_3.$$

$\bullet$	q_0	q_1
q_0	q_0	q_1
q_1	q_1	$-q_0$

$\bullet$	q_2	q_3
q_2	q_2	q_3
q_3	q_3	$-q_2$

$$Q = a_0q_0+a_1q_1+a_2q_2+a_3q_3 = \begin{bmatrix}a_0, & -a_1, & a_1, & -a_0\\ a_1, & a_0, & -a_0, & -a_1\\ a_3, & -a_2, & a_2, & -a_3\\ -a_2, & -a_3, & a_3, & a_2\end{bmatrix}$$

Fig. 13. Top, left: the decomposition of the matrix W_2 (Fig. 4) into 4 sparse matrices q_0, q_1, q_2, q_3. Top, right: multiplication tables for the pair q_0 and q_1 and for the pair q_2 and q_3. Bottom: the matrix representation of the system of 2-block U-complex numbers $Q = a_0q_0 + a_1q_1 + a_2q_2 + a_3q_3$.

$$e_0 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&0&0\\0&-1&0&0\end{bmatrix} \Leftrightarrow \begin{bmatrix}CC&0&0&0\\0&CG&0&0\\TC&0&0&0\\0&TG&0&0\end{bmatrix};\quad e_1 = \begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix} \Leftrightarrow \begin{bmatrix}0&CA&0&0\\CT&0&0&0\\0&TA&0&0\\TT&0&0&0\end{bmatrix}$$

Fig. 14. The conformity of each of non-zero entries in matrices e_0, e_1 from Fig. 5 with the corresponding doublet occupying the same position in the initial matrix $[C\ A;\ T\ G]^{(2)}$ (Fig. 3).

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
e_0 = {CC, CG, TC, TG}	00, 00, 10, 10	00, 01, 00, 01
e_1 = { TT, TA, CA, CT}	11, 11, 01, 01	00, 01, 01, 00
e_2 = {GG, GC, AG, AC}	00, 00, 10, 10	11, 10, 11, 10
e_3 = {AA, AT, GT, GA}	11, 11, 01, 01	11, 10, 10, 11

Fig. 15. The first column: representations of the conformity of numeric matrices e_0, e_1, e_2, e_3 (Fig. 5) and genetic doublets in the symbolic matrix $[C, A; T, G]^{(2)}$ (Fig. 3). The second and third columns: the complementarity indicators and the purin indicators for corresponding doublets.

Results of actions of the matrix operators Z_L (or Z_R) on a 4-dimensional vector $X = [x_0,x_1,x_2,x_3]$ depend on the side of the action: results of multiplication from right side $X\bullet Z_L$ and from left side $Z_L\bullet X$ differ and they have important properties concerning 2-dimensional subspaces of configurational 4-dimensional spaces (Figs. 10 and 11).

In the case of multiplication from the right side (Fig. 10), the possibility is revealed for a selective manage of vectors of 2-dimensional subspaces of configurational 4-dimensional spaces. The action $[x_0,x_1,x_2,x_3]\bullet Z_L$ gives the result that non-zero vectors exist only in the subspace (x_0,x_1), where they can be managed arbitrary by means of parameters a_0 and a_1 of the operator Z_L. If the parameters a_0 and a_1 are functions of time, then the vector $[x_0,x_1,x_2,x_3]\bullet Z_L$ moves along different trajectories in the plane (x_0,x_1) of the 4-dimensional space $[x_0,x_1,x_2,x_3]$. In other words, the corresponding 2-parametrical subsystem of the 4-parametrical system is running under the selective manage of the operator Z_L. It leads to valuable capabilities to model multi-parametrical biological systems and bioinformational spaces, where different subsystems and their configurational subspaces are managed independently. Our work (Petoukhov, 2017) shows that such block-hypercomplex operators can be generalized into 2^n-block-hypercomplex operators by means of their tensor multiplication with appropriate square matrices.

A similar situation of the selective manage is true for results of the action $[x_0,x_1,x_2,x_3]\bullet Z_R$ (Fig. 10): in this case non-zero vectors exist only in the subspace (x_2,x_3), where they can be managed arbitrary by means of parameters a_2 and a_3 of the operator Z_R. If the parameters a_2 and a_3 are functions of time, then the vector $[x_0,x_1,x_2,x_3]\bullet Z_R$ moves along different trajectories in the plane (x_2,x_3) of the 4-dimensional space $[x_0,x_1,x_2,x_3]$.

In the case of multiplication from the left side, the interesting effects of twinning appears (Fig. 11).

Fig. 11 shows that multiplication from the left side $Z_L\bullet[x_0,x_1,x_2,x_3]^T$ generates a new vector $[y_0,y_1,y_2,y_3]$ whose components y_0 and y_2 are identical to each other $(a_0x_0-a_1x_1)$ and whose components y_1 and y_3 are identical to each other up to sign: $y_1 = a_0x_1 + a_1x_0$ and $y_3 = -(a_0x_1 + a_1x_0)$ correspondingly. Such mathematical peculiarities of the multiplication from the left side $Z_L\bullet[x_0,x_1,x_2,x_3]^T$ can be called as effects of twinning. All components of the resulting vector $[y_0,y_1,y_2,y_3]$ in this case are independent on components x_2 and x_3 of the initial vector.

Analogically, multiplication from the left side $Z_R\bullet[x_0,x_1,x_2,x_3]^T$ generates a new vector $[y_0,y_1,y_2,y_3]$ whose components y_0 and y_2 are identical to each other $(a_2x_2-a_3x_3)$ and whose components y_1

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
q_0 = {CC, CG, AT, AA},	00, 00, 11, 11	00, 01, 10, 11
q_1 = {CA, CT, AC, AG},	01, 01, 10, 10	01, 00, 10, 11
q_2 = {GG, GC, TA, TT},	00, 00, 11, 11	11, 10, 01, 00
q_3 = {GT, GA, TC, TG}.	01, 01, 10, 10	10, 11, 00, 01

Fig. 16. The first column: representations of the conformity of numeric matrices q_0, q_1, q_2, q_3 (Fig. 13) and genetic doublets in the symbolic matrix $[C, A; T, G]^{(2)}$ (Fig. 3). The second and third columns: the complementarity indicators and the purin indicators for corresponding doublets.

$$D = a\bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\bullet \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

•	1	i
1	1	i
i	i	1

Fig. 17. The matrix representattion of split-complex numbers.

and y_3 are identical to each other up to sign: $-(a_2x_3+a_3x_2)$ and $a_2x_3+a_3x_2$ correspondingly. All components of the resulting vector $[y_0,y_1,y_2,y_3]$ are independent on components x_0 and x_1 of the initial vector.

Results of actions of the matrix operators $Z_2 = Z_L + Z_R$ on a 4-dimensional vector $X = [x_0,x_1,x_2,x_3]$ depend on the side of the action (Fig. 12) and they join together results of actions of separate operators Z_L and Z_R (see Figs. 10 and 11). The above described property of the selective manage of 2-dimensional subspaces of the 4-dimensional space is conserved in the case of multiplication by Z_2 from the right side. The described effect of twinning is also conserved in the case of multiplication by Z_2 from the left side.

One should note the existence of another decomposition of the Walsh-representation W_2 (Fig. 4) of the matrix of 16 doublets $[C, A; T, G]^{(2)}$ into sum of another set of 4 sparse matrices q_0, q_1, q_2, q_3 (Fig. 13). Each of its two sub-sets (q_0 and q_1) and (q_2 and q_3) is closed in relation to multiplication and corresponds again to the multiplication table of complex numbers (Fig. 13, right).

Correspondingly expressions $a_0q_0 + a_1q_1 = Q_L$ and $a_2q_2 + a_3q_3 = Q_R$ represent two new systems of complex numbers by analogy with the described case of the systems Z_L and Z_R. The total matrix $Q = Q_L + Q_R$ represents a new variant of 2-block U-complex numeric systems (Fig. 13). One can check that mathematical properties of both systems of 2-block U-complex numbers ($Z_L + Z_R$ and $Q_L + Q_R$) are similar.

One can note that the described decompositions (Fig. 5, 13) of the Walsh-representation W_2 have hidden regularities: they are connected with the «complementarity indicators» of 16 genetic doublets on the basis of numerations C = G = 0 and A = T = 1 of complementary DNA-pairs and also with the "purin indicators" on the basis of numerations of purins A = G = 1 and pyrimidines C = T = 0. Each of doublets is uniquely defined by its pair of such indicators. For example, only the doublet CG has its complementarity indicator 00 and its purin indicator 01. Each of sparse numeric matrices e_0, e_1, e_2, e_3 in Fig. 5 uniquely conforms to a sparse symbolic matrix produced from the matrix $[C\ A; T\ G]^{(2)}$ (Fig. 3). Fig. 14 explains it by the example of matrices e_0 and e_1.

By this conformity, numeric matrices e_0, e_1, e_2, e_3 (Fig. 5) can be written in the following symbolic forms (Fig. 15):

It is easy to see from Fig. 15 that the composition of the doublets related with e_0 and e_1 (CC, CG, TC, TG and TT, TA, CA, CT) and the composition of the doublets related with e_2 and e_3 (GG, GC, AG, AC and AA, AT, GT, GA) are complementary to each other. It means that – in this system of 2-block U-complex numbers – the algebraical complementary relations between two sub-systems of complex numbers ($e_0 + e_1$ and $e_2 + e_3$) can be confronted with the phenomenon of complementary relations between sequences of doublets in two filaments of the double helix of DNA.

The local-real units e_0 and e_2 have identical sets of binary complementarity indicators (00, 00, 10, 10); the same is true for the local-imaginary units e_1 and e_3 (11, 11, 01, 01). The complementarity indicators (00, 00, 10, 10) of the local-real units are inverse-symmetrical to the complementarity indicators (11, 11, 01, 01) of the local-imaginary units. One can also see symmetrical relations among sets of purin indicators (Fig. 15). All these features give aditional evidences in favor that molecular-genetic systems are connected with dyadic groups of binary numbers (Petoukhov, 2016b, 2017; Petoukhov, Petukhova, 2017). The phenomenon of strong and weak doublets is represented in the algebraic separation of the set of 16 doublets into 4 sub-sets (Fig. 15) in the following natural way. Each of local-real units e_0 and e_2 is connected with 3 strong doublets (CC, CG, TC and GG, GC, AC) and 1 weak doublet (TG and AG). In contrast, each of local-imaginary units e_1 and e_3 contains 3 weak doublets (TT, TA, CA and AA, AT, GA) and 1 strong doublet (CT and GT).

Analogically the set of sparse matrices q_0, q_1, q_2, q_3 (Fig. 13) has the following symbolic writing (Fig. 16).

Properties of complementarity and symmetries in this set q_0, q_1, q_2, q_3 (Fig. 16) coincide in many aspects with the described properties of the set e_0, e_1, e_2, e_3 in Fig. 15. Each of local units q_0,

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} = s_0+s_1+s_2+s_3$$

•	s_0	s_1
s_0	s_0	s_1
s_1	s_1	s_0

•	s_2	s_3
s_2	s_2	s_3
s_3	s_3	s_2

;

$$S = a_0s_0+a_2s_1+a_2s_2+a_3s_3 = \begin{bmatrix} a_0, & -a_0, & a_1, & -a_1 \\ a_2, & a_2, & -a_3, & -a_3 \\ a_1, & -a_1, & a_0, & -a_0 \\ -a_3, & -a_3, & a_2, & a_2 \end{bmatrix}$$

Fig. 18. Top: the decomposition of the Walsh-representation W_2 of the matrix $[C\ A; T\ G]^{(2)}$ of 16 doublets into 4 sparse matrices s_0, s_1, s_2, s_3. Bottom, left: multiplication tables of the set of matrices s_0 and s_1 and the set of matrices s_2 and s_3 coincide with the multiplication table of split-complex numbers in Fig. 15. Bottom, right: the matrix representation of 2-block U-split-complex numbers S.

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
s_0 = {CC, GC, CA, GA}	00, 00, 01, 01	00, 10, 01, 11
s_1 = {AC, TC, TA, AA}	10, 10, 11, 11	10, 00, 01, 11
s_2 = {GG, CG, GT, CT}	00, 00, 01, 01	11, 01, 10, 00
s_3 = {TG, AG, AT, TT}	10, 10, 11, 11	01, 11, 10, 00

Fig. 19. The first column: representations of the conformity of numeric matrices s_0, s_1, s_2, s_3 (Fig. 18) and genetic doublets in the symbolic matrix [C, A; T, G]$^{(2)}$ (Fig. 3). The second and third columns: the complementarity indicators and the purin indicators for corresponding doublets.

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = p_0+p_1+p_2+p_3$$

•	p_0	p_1
p_0	p_0	p_1
p_1	p_1	p_0

•	p_2	p_3
p_2	p_2	p_3
p_3	p_3	p_2

$$; \quad P = a_0p_0+a_2p_1+a_2p_2+a_3p_3 = \begin{bmatrix} a_0, & -a_2, & a_3, & -a_1 \\ a_1, & a_3, & -a_2, & -a_0 \\ a_0, & -a_2, & a_3, & -a_1 \\ -a_1, & -a_3, & a_2, & a_0 \end{bmatrix}$$

Fig. 20. Top: the decomposition of the Walsh-representation W_2 of the matrix [C A; T G]$^{(2)}$ of 16 doublets into 4 sparse matrices p_0, p_1, p_2, p_3. Bottom, left: multiplication tables of the set of matrices p_0 and p_1 and the set of matrices p_2 and p_3 coincide with the multiplication table of split-complex numbers in Fig. 17. Bottom, right: the matrix expression of 2-block U-split-complex numbers P.

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
p_0 = {CC, GG, AG, TC},	00, 00, 10, 10	00, 11, 11, 00
p_1 = {AA, TT, CT, GA},	11, 11, 01, 01	11, 00, 00, 11
p_2 = {AT, TA, CA, GT}	11, 11, 01, 01	10, 01, 01, 10
p_3 = {CG, GC, AC, TG}	00, 00, 10, 10	01, 10, 10, 01

Fig. 21. The first column: representations of the conformity of numeric matrices p_0, p_1, p_2, p_3 (Fig. 20) and genetic doublets in the symbolic matrix [C, A; T, G]$^{(2)}$ (Fig. 3). The second and third columns: the complementarity indicators and the purin indicators for corresponding doublets.

q_1, q_2 and q_3 is symmetrically connected with 2 strong doublets and 2 weak doublets. Principles of symmetries in matrix representations of structured alphabets of DNA and RNA are met on different steps of their algebraic analysis. Below we will again use this symbolic form of denotations of sparse matricies in other cases of decompositions of genetic matrices W_2 and W_3.

Let us turn now to another type of 2-dimensional numbers, which are known under the name «spit-complex numbers» (their synonyms are hyperbolic numbers, Lorentz numbers, double numbers, perplex numbers) (Split-complex number, 2017). Fig. 17 shows the usual (2×2)-matrix representation of split-complex numbers $D = a\bullet1 + b\bullet i$, the imaginary unit of which «i» satisfies to the condition $i^2 = +1$.

The Walsh-representation W_2 (Fig. 4) can be also decomposed into another set of 4 sparse matrices s_0, s_1, s_2, s_3, which represents the 2-block united-split-complex number S with unit values of its coordinates (Fig. 18).

The set of matrices s_0 and s_1 is closed under multiplication and defines the multiplication table of split-complex numbers (Fig. 18). The same is true for the set of matrices s_2 and s_3. Correspondingly the expression $S = a_0s_0 + a_2s_1 + a_2s_2 + a_3s_3$ represents the 4-parametrical system of 2-block united-split-complex numbers (Fig. 18). Here s_0 and s_2 are local-real units and s_1 and s_3 are local-imaginary units (by analogy with the described case of local-real and local-imaginary units in 2-block U-complex numbers (Fig. 5)).

Fig. 19 shows the symbolic writing of these sparse matrices s_0, s_1, s_2, s_3 with their relations of the complementarity among doublets and also of symmetries in each of sets of complementarity indicators and purin indicators. Here each of local units s_0 and s_1 is connected with 2 strong doublets (CC, GC and AC, TC) and 2 weak doublets (CA, GA and TA, AA). The local unit s_2 is connected with 4 strong doublets in contrast to the local unit s_3, which is connected with 4 weak doublets.

The Walsh-representation W_2 (Fig. 4) can be decomposed into one more set of 4 sparse matrices p_0, p_1, p_2, p_3, which leads to another system of 2-block U-split-complex numbers (Fig. 20).

The set of matrices p_0 and p_1 is closed under multiplication and defines the multiplication table of split-complex numbers (Fig. 20). The same is true for the set of matrices p_2 and p_3. Correspondingly the expression $P = a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3$ represents the 4-parametrical system of 2-block united-split-complex numbers (Fig. 20). Here p_0 and p_2 are local-real units and p_1 and p_3 are local-imaginary units.

Fig. 21 shows the symbolic writing of these sparse matrices p_0, p_1, p_2, p_3. Here each of p_0 and p_3 is connected with 3 strong doublets (CC, GG, TC and CG, GC, AC) and 1 weak doublet (AG and TG). In contrast, each of p_1 and p_2 is connected with 3 weak doublets (AA, TT, GA and AT, TA, CA) and 1 strong doublet (CT and TG).

Fig. 15, 16, 19, 21 show that all 4 decompositions of the genetic matrix W_2, which lead to the described systems of 2-block U-numbers, have hidden relations with dyadic group of binary numbers and with dyadic-shift matrices; this fact is inter-

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
q_{0perm} = {CC, GC, TA, AA},	00, 00, 11, 11	00, 10, 01, 11
q_{1perm} = {AC, TC, CA, GA},	10, 10, 01, 01	10, 00, 01, 11
q_{2perm} = {GG, CG, AT, TT},	00, 00, 11, 11	11, 01, 10, 00
q_{3perm} = {TG, AG, CT, GT}.	10, 10, 01, 01	01, 11, 00, 10

$$q_{0perm} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad q_{1perm} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad q_{2perm} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad q_{3perm} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Fig. 22. Top: sub-sets of doublets, which are received from sub-sets q_0, q_1, q_2, q_3 (Fig. 13, 16) in the result of the permutation 2-1 of positions in all their doublets. Bottom: new sparse matrices q_{0perm}, q_{1perm}, q_{2perm}, q_{3perm} appear.

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
q_{0rev} = {GG, CG, AT, TT},	00, 00, 11, 11	11, 01, 10, 00
q_{1rev} = {TG, AG, GT, CT},	10, 10, 01, 01	01, 11, 10, 00
q_{2rev} = {CC, GC, TA, AA},	00, 00, 11, 11	00, 10, 01, 11
q_{3rev} = {AC, TC, GA, CA}.	10, 10, 01, 01	10, 00, 11, 01

Fig. 23. Sub-sets of doublets, which are received from sub-sets q_0, q_1, q_2, q_3 (Fig. 13, 16) in the result of the replacement of all doublets by their reverse complements.

Sparse matrices	Complementarity indicators (C=G=0, A=T=1)	Purin indicators (C=T=0, A=G=1)
p_{0perm} = {CC, GG, GA, CT},	00, 00, 01, 01	00, 11, 11, 00
p_{1perm} = {AA, TT, TC, AG},	11, 11, 10, 10	11, 00, 00, 11
p_{2perm} = {TA, AT, AC, TG}	11, 11, 10, 10	01, 10, 10, 01
p_{3perm} = {GC, CG, CA, GT}	00, 00, 01, 01	10, 01, 01, 10

$$p_{0perm} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_{1perm} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad p_{2perm} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad p_{3perm} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 24. Top: sub-sets of doublets, which are received from sub-sets p_0, p_1, p_2, p_3 (Fig. 20, 21) in the result of the permutation 2-1 of positions in all their doublets. Bottom: new sparse matrices p_{0perm}, p_{1perm}, p_{2perm}, p_{3perm} appear.

esting from the standpoint of the concept of geno-logical coding (Petoukhov, 2016b, 2017; Petoukhov and Petoukhov, 2017).

The described algebraic analysis of genetic matrix $[C, A; T, G]^{(2)}$ (Fig. 3) reveals that the alphabet of 16 doublets can be represented by different ways as a composition of 4 sub-alphabets connected with systems of 2-block U-complex or U-split-complex numbers (Figs. 15, 16, 19, 21). Each of the sub-alphabets contains 4 doublets.

4. U-numerical systems and permutations in the DNA-alphabet of doublets

In studies of genetic coding, a special attention is paid to cyclic codes connected with cyclic shifts or cyclic permutations of elements in multi-block sequences (Arquès and Michel, 1996; Fimmel et al., 2014, 2015, 2016; Fimmel and Strüngmann, 2015; Michel and Seligmann, 2014). Permutations play an important role in digital signal processing; for example, bit-reversal permutations are connected, in particularly, with quasi-holographic models, noise-immunity coding and with algorithms of fast Fourier transform (Gold and Rader, 1969; Karp, 1996; Shishmintsev, 2012; Yang et al., 2013). Taking this into account we show in this Section an interesting behavior of the described U-numerical systems under various permutations in the alphabet of 16 doublets represented by the matrix $[C, A; T, G]^{(2)}$ (Fig. 3). Our results attest to the conformity of the algebraic properties of the U-numerical systems with phenomenological properties of the DNA-alphabets and with the complementary device of the double DNA-helix.

For example let us turn to the set of 2-block U-complex numbers with its local units q_0, q_1, q_2, q_3 (Figs. 13 and 16). The permutation of positions in each of doublets according to the order 2-1 (instead of the initial order 1–2) transforms the symbolic writing of q_0, q_1, q_2, q_3 into the symbolic writing (Fig. 22, top) of new matrices q_{0perm}, q_{1perm}, q_{2perm}, q_{3perm}. Fig. 22 shows also these new numerical matrices, entries of which correspond to locations of appropriate doublets in the matrix [C, A; T, G]$^{(2)}$ (Fig. 3). This permutation transformation conserves the described complementarity among doublets in the sub-sets.

Unexpectedly in this new set of sparse matrices each of two pairs of matrices (q_{0perm}, q_{1perm}) and (q_{2perm}, q_{3perm}) is again closed under multiplication and defines the multiplication table of split-complex numbers (Fig. 17). It means that this permutation transformation in doublets has a non-trivial algebraic meaning and leads to the new system of 2-block U-split-complex numbers with new kinds of their local-real and local-imaginary units. Such facts also attest to the conformity of the algebraic properties of the U-numerical systems with phenomenological properties of the DNA-alphabets and with the complementary device of the double DNA-helix.

Let us do an additional computational experiment devoted to the so-called reverse complements. Many authors devote their works to the reversal complements in a relation with the second law of E. Chargaff, numerous cases of palindromes in DNA-sequences, etc. (Bell and Forsdyke, 1999; Forsdyke, 1995; Qi and Cuticchia, 2001; Mitchell and Bride, 2006; Perez, 2010). The reverse complement is obtained by transposing each nucleotide into its complementary nucleotide (A → T, T → A, C → G, G → C), and then reversing the sequence of nucleotides. For example, the doublet CA has its reverse complement TG. If we replace each of doublets in the system q_0, q_1, q_2, q_3 by its reverse compliment, a set of matrices q_{0rev}, q_{1rev}, q_{2rev}, q_{3rev} arises, which are represented in their symbolic form in Fig. 23. One can see from a comparison of Fig. 22 and Fig. 23 that this set repeats the set of matrices q_{0perm}, q_{1perm}, q_{2perm}, q_{3perm} from Fig. 22 but in another order: $q_{0rev} = q_{2perm}$, $q_{1rev} = q_{3perm}$, $q_{2rev} = q_{0perm}$, $q_{3rev} = q_{1perm}$. In other words, in this kind of permutations in all doublets the same system of 2-block U-split-complex numbers arises, where previous two split-complex parts (q_{0perm}, q_{1perm}) and (q_{2perm}, q_{3perm}) just swapped their places.

Similar results can be demonstrated in the case of the system of 2-block U-split-complex numbers with basic matrices p_0, p_1, p_2, p_3 (Figs. 20 and 21). The permutation of positions in each of doublets according to the order 2-1 transforms the symbolic writing of p_0, p_1, p_2, p_3 into the symbolic writing (Fig. 24, top) of new matrices p_{0perm}, p_{1perm}, p_{2perm}, p_{3perm}. Fig. 24 shows also these new numerical matrices, entries of which correspond to locations of appropriate doublets in the matrix [C, A; T, G]$^{(2)}$ (Fig. 3).

In this new set of sparse matrices each of two pairs of matrices (p_{0perm}, p_{1perm}) and (p_{2perm}, p_{3perm}) is again closed under multiplication and defines the multiplication table of split-complex numbers (Fig. 17). It means that this permutation transformation in doublets has a non-trivial algebraic meaning and leads to the new system of 2-block U-split-complex numbers with new kinds of their local-real and local-imaginary units.

5. United-complex numbers and sub-alphabets of 16 doublets

The described algebraic analysis of the matrix [C, A; T, G]$^{(2)}$, which represents the DNA-alphabet of 16 doublets, draws attention to the following fact, which we uses in developing "personalized algebra-logical genetics". The alphabet of 16 doublets contains quaternary sub-alphabets of different kinds, which are connected with the systems of U-complex numbers and U-split-complex numbers ((Figs. 15, 16, 19, 21). In each of these sub-alphabets, all 16 doublets are separated in 4 groups of equivalence by special traits; inside each of 4 groups all its 4 doublets is denoted identically by means of one of members of the dyadic group of 2-bit binary numbers: 00, 01, 10, 11. Let us consider these quaternary sub-alphabets more attentively.

Fig. 15 shows that – from the standpoint of the U-numerical system of matrices e_0, e_1, e_2, e_3 – the alphabet of 16 doublets contains the quaternary sub-alphabet on the basis of the following traits of equivalency among doublets:

- all 4 doublets, which are associated with the local-real unit e_0, are considered as equivalent each other and they can be conditionally numerated by 2-bit binary number 00_e (CC = CG = TC = TG = 00_e) where the index "e" marks that these doublets have such binary meaning only in this sub-alphabet;
- all 4 doublets, which are associated with the local-imaginary unit e_1, are considered as equivalent each other and they can be conditionally numerated by 2-bit binary number 01_e (TT = TA = CA = CT = 01_e);
- all 4 doublets, which are associated with the local-real unit e_2, are considered as equivalent each other and they can be conditionally numerated by 2-bit binary number 10_e (GG = GC = AG = AC = 10_e);
- all 4 doublets, which are associated with the local-imaginary unit e_3, are considered as equivalent each other and they can be conditionally numerated by 2-bit binary number 11_e (AA = AT = GT = GA = 11_e).

Analogically Fig. 16 shows the second kind of quaternary sub-alphabets, which is produced by the separation of the same set of 16 doublets into other 4 groups of equivalence based on their association with matrices q_0, q_1, q_2, q_3:

- CC = CG = AT = AA = 00_q;
- CA = CT = AC = AG = 01_q;
- GG = GC = TA = TT = 10_q;
- GT = GA = TC = TG = 11_q (the index «q» marks that these doublets have such binary meaning only in this sub-alphabet).

Analogically Fig. 19 shows the third kind of quaternary sub-alphabets, which is produced by the separation of the same set of 16 doublets into other 4 groups of equivalence based on their association with matrices s_0, s_1, s_2, s_3:

- CC = GC = CA = GA = 00_s;
- AC = TC = TA = AA = 01_s;
- GG = CG = GT = CT = 10_s;
- TG = AG = AT = TT = 11_s (the index «s» marks that these doublets have such binary meaning only in this sub-alphabet).

Analogically Fig. 21 shows the fourh kind of quaternary sub-alphabets, which is produced by the separation of the same set of 16 doublets into other 4 groups of equivalence based on their association with matrices p_0, p_1, p_2, p_3:

- CC = GG = AG = TC = 00_p;
- AA = TT = CT = GA = 01_p;
- AT = TA = CA = GT = 10_p;
- CG = GC = AC = TG = 11_p (the index «p» marks that these doublets have such binary meaning only in this sub-alphabet).

Reading of any of DNA-sequences can depend on a choice among these sub-alphabets. For example, the DNA-sequence CG AA TG TT CA TC is read by the following ways:

- by means of the first sub-alphabet as $00_e11_e00_e01_e01_e00_e$;
- by means of the second sub-alphabet as $00_g00_g11_g10_g01_g11_g$, etc.

Various kinds of sub-alphabets of doublets, triplets and other n-plets, which are connected with appropriate systems of U-hypercomplex numbers, can be briefly called in a general case as

$$W_3 = \begin{bmatrix} 1,1,-1,-1,1,1,-1,-1 \\ 1,1,-1,-1,1,1,-1,-1 \\ 1,1,1,1,-1,-1,-1,-1 \\ 1,1,1,1,-1,-1,-1,-1 \\ 1,1,-1,-1,1,1,-1,-1 \\ 1,1,-1,-1,1,1,-1,-1 \\ -1,-1,-1,-1,1,1,1,1 \\ -1,-1,-1,-1,1,1,1,1 \end{bmatrix} = \begin{bmatrix} 1,0,0,0,0,0,0,0 \\ 1,0,0,0,0,0,0,0 \\ 0,0,1,0,0,0,0,0 \\ 0,0,1,0,0,0,0,0 \\ 1,0,0,0,0,0,0,0 \\ 1,0,0,0,0,0,0,0 \\ 0,0,-1,0,0,0,0,0 \\ 0,0,-1,0,0,0,0,0 \end{bmatrix} + \begin{bmatrix} 0,0,-1,0,0,0,0,0 \\ 0,0,-1,0,0,0,0,0 \\ 1,0,0,0,0,0,0,0 \\ 1,0,0,0,0,0,0,0 \\ 0,0,-1,0,0,0,0,0 \\ 0,0,-1,0,0,0,0,0 \\ -1,0,0,0,0,0,0,0 \\ -1,0,0,0,0,0,0,0 \end{bmatrix} + \begin{bmatrix} 0,1,0,0,0,0,0,0 \\ 0,1,0,0,0,0,0,0 \\ 0,0,0,1,0,0,0,0 \\ 0,0,0,1,0,0,0,0 \\ 0,1,0,0,0,0,0,0 \\ 0,1,0,0,0,0,0,0 \\ 0,0,0,-1,0,0,0,0 \\ 0,0,0,-1,0,0,0,0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0,0,0,-1,0,0,0,0 \\ 0,0,0,-1,0,0,0,0 \\ 0,1,0,0,0,0,0,0 \\ 0,1,0,0,0,0,0,0 \\ 0,0,0,-1,0,0,0,0 \\ 0,0,0,-1,0,0,0,0 \\ 0,-1,0,0,0,0,0,0 \\ 0,-1,0,0,0,0,0,0 \end{bmatrix} + \begin{bmatrix} 0,0,0,0,1,0,0,0 \\ 0,0,0,0,1,0,0,0 \\ 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,1,0,0,0 \\ 0,0,0,0,1,0,0,0 \\ 0,0,0,0,0,0,1,0 \\ 0,0,0,0,0,0,1,0 \end{bmatrix} + \begin{bmatrix} 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,-1,0,0,0 \\ 0,0,0,0,-1,0,0,0 \\ 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,0,0,-1,0 \\ 0,0,0,0,1,0,0,0 \\ 0,0,0,0,1,0,0,0 \end{bmatrix} + \begin{bmatrix} 0,0,0,0,0,1,0,0 \\ 0,0,0,0,0,1,0,0 \\ 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,1,0,0 \\ 0,0,0,0,0,1,0,0 \\ 0,0,0,0,0,0,0,1 \\ 0,0,0,0,0,0,0,1 \end{bmatrix} + \begin{bmatrix} 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,-1,0,0 \\ 0,0,0,0,0,-1,0,0 \\ 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,0,0,-1 \\ 0,0,0,0,0,1,0,0 \\ 0,0,0,0,0,1,0,0 \end{bmatrix}$$

Fig. 25. The decomposition of the matrix W_3 (Fig. 4) into sum of 8 sparse matrices: $W_3 = r_0 + r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7$.

•	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_0	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_1	r_1	$-r_0$	r_3	$-r_2$	$-r_5$	r_4	$-r_7$	r_6
r_2	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_3	r_1	$-r_0$	r_3	$-r_2$	$-r_5$	r_4	$-r_7$	r_6
r_4	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_5	$-r_1$	r_0	$-r_3$	r_2	r_5	$-r_4$	r_7	$-r_6$
r_6	r_0	r_1	r_2	r_3	r_4	r_5	r_6	r_7
r_7	$-r_1$	r_0	$-r_3$	r_2	r_5	$-r_4$	r_7	$-r_6$

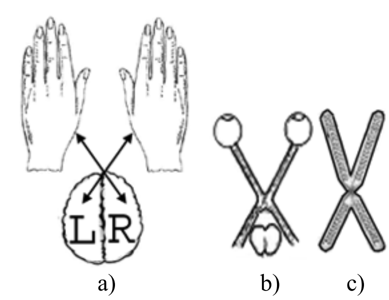

Fig. 26. Left: the cross-shaped multiplication table of matrices r_0, r_1,. . ., r_7 from Fig. 25. Right: the cross-shaped schemes of some morpho-functional structures in human organism: a) the cross-shaped connections of brain hemispheres with the left and the right halves of a human body; b) the cross-shaped structure of optic nerves from eyes in brain; c) the cross-shaped structure of a chromosome.

"U-numerical sub-alphabets". As known, each of complex computer systems with different informational channels and many levels (like as the World Wide Web) must have many languages, which can use binary numbers, for their effective functioning (Manin and Manin, 2017).

The example of the World Wide Web shows that such a system of multi-level and multi-channel computer science – by using many languages for a chain transfer of information from one level or one node to neighboring levels or nodes – is capable of providing an information transmission in a noise-immunity manner from the lowest levels to very remote higher levels. The author thinks that a similar multi-level and multi-channel system – with using many languages for informational exchanges between neighboring levels and nodes – is a basis in genetics for an information transmission in a noise-immunity manner from the lowest molecular levels of DNA and RNA to the highest levels of living bodies. Correspondingly the author puts forward the following "hypothesis of genetic multi-linguistics" to explain, in particular, noise-immunity properties of the genetic system:

- Many different reading systems exist in molecular genetics to take information from DNA-sequences, which simultaneously carry many parallel informational messages on different languages, where each of messages can contain quite individual sense or instructions; these original molecular messages are processed and transmitted in a noise-immunity manner via many channels to the highest levels of living bodies by means of using many languages in chains of informational exchanges between neighboring levels and nodes;
- Each of such U-numerical sub-alphabets is a basis of one of appropriate genetic languages, which are not absolutely separated but are interrelated each with other via certain algebraic interrelations of their sub-alphabets.

This hypothesis is supported by various facts of conformities between algebraic features of U-hypercomplex numbers and phenomenological structures of molecular genetics including structures of DNA- and RNA-alphabets. We think that one of the main reasons of very special structuring of these genetic alphabets is related with the important task of providing this genetic multi-linguistics.

6. United-hypercomplex numbers and the genetic matrix of 64 triplets

Let us consider now the Walsh-representation W_3 (Fig. 4) of the matrix of 64 triplets $[C, A; T, G]^{(3)}$ (Fig. 3). Fig. 25 shows the

Sparse matrices	Amino acids from Fig. 3
r_0 = {CCC, CGC, CCT, CGT, TCC, TGC, TCT, TGT}	**Pro**, **Arg**, **Pro**, **Arg**, **Ser**, Cys, **Ser**, Cys
r_1 = {CAC, CTC, CAT, CTT, TAC, TTC, TAT, TTT}	His, **Leu**, His, **Leu**, Tyr, Phe, Tyr, Phe
r_2 = {CCG, CGG, CCA, CGA, TCG, TGG, TCA, TGA}	**Pro**, **Arg**, **Pro**, **Arg**, **Ser**, Trp, **Ser**, Trp
r_3 = {CAG, CTG, CAA, CTA, TAG, TTG, TAA, TTA}	Gln, **Leu**, Gln, **Leu**, Stop, Leu, Stop, Leu
r_4 = {GCC, GGC, GCT, GGT, ACC, AGC, ACT, AGT}	**Ala**, **Gly**, **Ala**, **Gly**,**Thr**, Ser, **Thr**, Ser
r_5 = {GAC, GTC, GAT, GTT, AAC, ATC, AAT, ATT}	Asp, **Val**, Asp, **Val**, Asn, Ile, Asn, Ile
r_6 = {GCG, GGG, GCA, GGA, ACG, AGG, ACA, AGA}	**Ala**, **Gly**, **Ala**, **Gly**, **Thr**, Stop, **Thr**, Stop
r_7 = {GAG, GTG, GAA, GTA, AAG, ATG, AAA, ATA}	Glu, **Val**, Glu, **Val**, Lys, Met, Lys, Met

Fig. 27. Left: the representation of the conformity of numeric matrices $r_0, r_1, \ldots, r_7$ (Fig. 25) and genetic triplets in the symbolic matrix $[C, A; T, G]^{(3)}$ (Fig. 3). Right: sets of amino acids and stop-signals (from Fig. 3) for each of 8 local units r_i of the system of 4-block U-complex numbers (Fig. 25). Each of amino acids, which is encoded by a riplet with a strong root (such triplets are mark by black color in the matrix in Fig. 3), is denoted by bold and underlined.

$$\mathbf{W_2 = W_{20}+W_{21}} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

Fig. 28. The matrix W_2 is the sum of two matrices W_{20} and W_{21}.

decomposition of the matrix W_3 into sum of 8 sparse matrices r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7.

Here each of the following 4 pairs of matrices – (r_0, r_1), (r_2, r_3), (r_4, r_5), (r_6, r_7) – is closed under multiplication and defines the multiplication table of complex numbers (Fig. 1). Correspondingly the following expression represents the 8-parametrical system of 4-block U-complex numbers: $a_0r_0 + a_1r_1 + a_2r_2 + a_3r_3 + a_4r_4 + a_5r_5 + a_6r_6 + a_7r_7$. Algebraic, symmetrical and permutation properties of this system are similar in many aspects to the above-described properties of systems of 2-block U-complex numbers.

The total set of matrices $r_0, r_1, \ldots, r_7$ is closed under multiplication and corresponds to the multiplication table in Fig. 26.

This multiplication table also has a cross-shaped structure with elements of a fractal plexus in relative locations of sets of basic units:

- two (4×4)-quadrants along its main diagonal contain all (2×2)-sub-quadrants, structure of which coincides with the multiplication table of basic elements of complex numbers; in opposite to this, two (4×4)-quadrants along its second diagonal contain all (2×2)-sub-quadrants, structure of which coincides with the cross-shaped matrix representation of complex numbers $[a, b; -b, a]$ (Fig. 1);
- the main diagonal of the (8×8)-table contains only cells with local-real units r_0, r_2, r_4, r_6; in opposite to this, its second diagonal contains only cells with local-imaginary units r_1, r_3, r_5, r_7;
- inside of each of the four (4×4)-quadrants its main diagonal contains only cells with local-real units; in opposite to this, its second diagonal contains only cells with local-imaginary units;
- inside of each of the sixteen (2×2)-sub-quadrants its main diagonal also contains only cells with one of the local-real units; in opposite to this, its second diagonal contains only cells with one of the local-imaginary units.

On the side we note that cross-shaped structures of the genetic matrices $[C A; T G]^{(2)}$ and $[C A; T G]^{(3)}$ (Fig. 3) and also of the multiplication tables of U-complex numbers (Figs. 8 and 26) are associated with known phenomena of inherited cross-shaped structures of physiological objects (see examples in Fig. 26, right):

- the cross-shaped connection between the hemispheres of human brain and the halves of human body managed by them: the left hemisphere serves the right half of the body and the right hemisphere serves the left half of the body (Annett, 1985, 1992; Gazzaniga, 1995; Hellige, 1993);
- The system of optic cranial nerves from two eyes possesses the cross-shaped structures as well: the optic nerves transfer information about the right half of field of vision into the left hemisphere of brain, and information about the left half of field of vision into the right hemisphere. The same is held true for the hearing system (Penrose, 1989; Chapter 9). One can suppose that these inherited physiological phenomena are connected with genetic cross-shaped structures, which participate in providing noise-immunity properties of genetic systems;
- cross-shaped chromosomes, etc.

This system of 4-block U-complex numbers (Fig. 25) gives the special separation of the whole set of 64 triplets into 8 sub-sets, each of whose members is connected with one of these matrices $r_0, r_1, \ldots, r_7$. Fig. 27 shows the symbolic writing of this 8 sub-sets, each member of which contains 8 triplets.

Fig. 27 additionally shows remarkable symmetries in the natural distribution of 20 amino acids and stop-codons among the 8 local units r_i of this system of 4-block U-complex numbers: each of local-real units r_0, r_2, r_4, r_6 is associated with such 3 amino acids, each of which is encoded by a triplet of a strong root (these amino acids are marked by bold and underlined in Fig. 27), and with 1 amino acid encoded by a triplet of a weak root; it is just the opposite to the case of local-imaginary units r_1, r_3, r_5, r_7, each of which is associated with such 3 amino acids, each of which is encoded by a triplet of a weak root, and with 1 amino acid encoded by a triplet of a strong root. As we now understand, one of examples of systems of 2-block U-quaternion numbers (non-Hamiltonian quaternions) is also the genetic Yin-Yang-algebra described early in (Petoukhov, 2008; Petoukhov and He, 2010).

One should remind that each of triplets in the sub-sets (Fig. 27) is uniquily defined by its two binary indicators: the complementarity indicator and the purin indicator. For example, only the triplet CGA has simultaneously the complementarity indicator 001 and the purin indicator 011. By analogy with the described quaternary sub-alphabets of doublets, these 8 units r_i of the system of 4-block U-complex numbers (Fig. 27) allow introducing a corresponding octal sub-alphabet of the DNA-alphabet of 64 triplets. In this octal sub-alphabet its 8 members, each of which contains 8 triplets, can be binary numerated by means of members of the dyadic group of 3-bit binary numbers 000_r, 001_r, 010_r, 011_r, 100_r, 101_r, 110_r,

$$\begin{bmatrix} 1 & -1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 1\end{bmatrix} = \begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix} + \begin{vmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix} + \begin{vmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix} + \begin{vmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{vmatrix} = d_0+d_1+d_2+d_3$$

•	d_0	d_1
d_0	d_0	d_1
d_1	d_1	$-d_0$

•	d_2	d_3
d_2	d_2	d_3
d_3	d_3	$-d_2$

Fig. 29. Top: the decomposition of the matrix W_{20} from Fig. 28: $W_{20} = d_0 + d_1 + d_2 + d_3$. Bottom: multiplication tables for the pair d_0 and d_1 and for the pair d_2 and d_3 coincide with the multiplication table of complex numbers.

$V_L = a_0d_0+a_1d_1 =$ $[a_0, -a_1, 0, 0$ $a_1, a_0, 0, 0$ $0, 0, 0, 0$ $0, 0, 0, 0]$	$V_R = a_2d_2+a_3d_3 =$ $[0, 0, 0, 0$ $0, 0, 0, 0$ $0, 0, a_2, -a_3$ $0, 0, a_3, a_2]$	$V = V_L+V_R = a_0d_0+a_1d_1+a_2d_2+a_3d_3 =$ $[a_0, -a_1, 0, 0$ $a_1, a_0, 0, 0$ $0, 0, a_2, -a_3$ $0, 0, a_3, a_2]$

$V_L^{-1} = (a_0^2 + a_1^2)^{-1} \bullet$ $(a_0d_0-a_1d_1)$; $V_LV_L^{-1}=V_L^{-1}V_L=d_0$, where d_0 – the local-identity matrix for V_L.	$V_R^{-1} = (a_2^2 + a_3^2)^{-1}\bullet$ $(a_2d_2-a_3d_3)$; $V_RV_R^{-1}=V_R^{-1}V_R=d_2$, where d_2 – the local-identity matrix for V_R.	$V^{-1} = (a_0^2+a_1^2)^{-1}\bullet(a_0d_0-a_1d_1) + (a_2^2+a_3^2)^{-1}\bullet(a_2d_2-a_3d_3)$ $= V_L^{-1} + V_R^{-1}$. $VV^{-1} = V^{-1}V = E$, where $E = [1,0,0,0; 0,1,0,0; 0,0,1,0; 0,0,0,1]$ – the identity matrix.

Fig. 30. Upper row: the system of diagonal 2-block U-complex numbers V, which unites systems of complex numbers V_L and V_R. Lower row: expressions for inverse matrices V_L^{-1}, V_R^{-1} and V^{-1}.

111_r (here the index «r» marks that these triplets have such binary meaning only in this octal sub-alphabet). For example:

- all 8 triplets, which are associated with the local-real unit r_0, are considered as equivalent each other and they can be conditionally numerated by 3-bit binary number 000_r (CCC = CGC = CCT = CGT = TCC = TGC = TCT = TGT = 000_r);
- all 8 triplets, which are associated with the local-imaginary unit r_1, are considered as equivalent each other and they can be conditionally numerated by 3-bit binary number 001_r (CAC = CTC = CAT = CTT = TAC = TTC = TAT = TTT = 001_r), etc.

Let us add that the special mosaics of the genetic matrices in Fig. 3 are connected with the following non-trivial properties of their Walsh-representations W_2 and W_3 (Fig. 4). The exponentiation of the matrix W_2 in the second power doubles it without changing its structure: $W_2{}^2 = 2W_2$. The exponentiation of the matrix W_3 in the second power quadruples it without changing its structure: $W_3{}^2 = 4W_3$. Accordingly, under such operations the algebraic properties of these matrices remain unchanged, including their connections with the systems of U-numbers and with the selective manage of subspaces. This doubling of the matrix W_2 causes association with the doubling of somatic cells and their genetic material under mitosis, and the quadruple of the matrix W_3 causes association with quadruple of gametes under meiosis. From the modeling standpoint these algebraic properties of the mosaic matrices of 16 doublets and 64 triplets can be considered as one of reasons of the biological phenomenon of the black-and-white separation in the sets of doublets and triplets (Figs. 3 and 4).

7. Diagonal U-hypercomplex numbers

Whether such systems of U-complex numbers exist, which possess operations of commutative multiplication and division? This Section gives positive answer on the question and shows systems with these operations and addiionally with divisors of zero. These systems were revealed in the result of our matrix-algebraical study of genetic matrices and their Walsh-representations W_2 and W_3 (Fig. 4).

In the matrix W_2 (Fig. 4) the mosaic of two quadrants along the main diagonal is mirror-antisymmetric to the mosaic of two quadrants along the second diagonal in relation to the middle vertical line. The operation of the mirror-antisymmetric transformation is given by the matrix $J = [0,0,0,-1; 0,0,-1,0; 0,-1,0,0; -1, 0,0,0]$, the square of which J^2 is equal to the identical matrix E: $J^2 = E$. Fig. 28 illustrates that the matrix W_2 is the sum of 2 sparse matrices W_{20} and W_{21} of diagonal types, non-zero parts of which coincide correspondingly with both quadrants along the main diagonal and with both quadrants along the second diagonal.

The mirror-antisymmetric transformation J transforms the matrix W_{21} into the matrix W_{20}. Correspondingly $W_2 = W_{20} + W_{21} = W_{20} + J\bullet W_{20} = W_{20}\bullet(E+J)$, where $E+J$ is the split-complex number with unit coordinates since the set of matrices E and J is closed under multiplication and its multiplication table coincides with the muliplication table of split-complex numbers. The matrix W_{20} is 2-block U-complex number of diagonal type with identity coordinates from the standpoint of its decomposition into 4 sparse matrices d_0, d_1, d_2, d_3 (Fig. 29).

The pair of matrices d_0 and d_1 forms the set, which is closed under multiplication and which defines the multiplication table of complex numbers (Fig. 29, bottom). The same is true for the pair of matrices d_2 and d_3. These basis elements d_0, d_1, d_2, d_3 define the following view of two systems of complex numbers V_L and V_R represented by (4×4)-matrices: $V_L = a_0d_0 + a_1d_1$ and $V_R = a_2d_2 + a_3d_3$. Fig. 30 shows the corresponding general view of the system of 2-block diagonal U-complex numbers $V = a_0d_0 + a_1d_1 + a_2d_2 + a_3d_3$ having 4 parameters (a_0, a_1, a_2, a_3). One should mention that the whole set of matrices d_0, d_1, d_2, d_3 is not closed under multiplication and it does not represent any of hypercomplex numbers.

The classical identity matrix E = [1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1] is absent in the set of matrices V_L and V_R, where – beside this – each matrix has zero determinant (Fig. 30). Consequently the notion of the inverse matrix V_L^{-1} or V_R^{-1} can't be defined in relation to the identity matrix E. But we analyze not the complete set of (4×4)-matrices but limited special sets of matrices V_L and V_R. The system V_L has the matrix d_0 (Fig. 29), which possesses all properties of its identity matrix for any matrix V_L since $d_0V_L = V_Ld_0 = V_L$ and $d_0^2 = d_0$. In the frame of the system of matrices V_L, where locally the matrix d_0 plays the role of the identity matrix (the local-identity matrix), one can define – for any non-zero matrix V_L – the inverse matrix V_L^{-1} in relation to the matrix d_0 on the basis of equations: $V_LV_L^{-1} = V_L^{-1}V_L = d_0$. Such inverse matrix is defined by the following expression: $V_L^{-1} = (a_0^2 + a_1^2)^{-1}\bullet(a_0d_0\text{-}a_1d_1)$ (Fig. 30).

By analogy, the system of matrices V_R has the matrix d_2 (Fig. 29), which possesses all properties of the identity matrix for any matrix V_R since $d_2V_R = V_Rd_2 = V_R$ and $d_2^2 = d_2$. In the frame of the system of matrices V_R, where the matrix d_2 is the local-identity matrix, one can define – for any non-zero matrix V_R – the inverse matrix V_R^{-1} in relation to the matrix d_2 on the base of equations: $V_RV_R^{-1} = V_R^{-1}V_R = d_2$.

Any of non-zero 2-block diagonal U-complex numbers $V = V_L + V_R = a_0d_0 + a_1d_1 + a_2d_2 + a_3d_3$ has its inverse number V^{-1} of the same type in relation to the ordinary identity matrix E = [1,0,0,0; 0,1,0,0; 0,0,1,0; 0,0,0,1]: $VV^{-1} = V^{-1}V = E$, where $V^{-1} = (a_0^2 + a_1^2)^{-1}\bullet(a_0d_0\text{-}a_1d_1) + (a_2^2 + a_3^2)^{-1}\bullet(a_2d_2\text{-}a_3d_3) = V_L^{-1} + V_R^{-1}$. In other words, in this system the inverse U-number V^{-1} is equal to the sum of inverse numbers V_L^{-1} and V_R^{-1}.

The system of 2-block diagonal U-complex numbers V has operations of addition, subtraction, commutative multiplication and division (division is defined by multiplication with the inverse diagonal U-numbers V^{-1}). This system has divisors of zero; it means that the system has non-zero matrices, the product of which gives zero. Indeed, the product of non-zero matrices V_L and V_R is equal to zero matrix: $V_LV_R = V_RV_L = 0$. (If split-complex numbers are used in both diagonal blocks of the matrix V in Fig. 29 instead of complex numbers, the system of diagonal 2-block U-split-complex numbers arises, which also possesses operations of addition, subtraction, commutative multiplication and division and which has divisors of zero).

Extensions of diagonal 2-block U-complex numbers into diagonal 2^n-block U-complex numbers can be easily done by means of tensor multiplication $[1\ 0;\ 0\ 1]^{(n)}\otimes[1,-1;\ 1,1]$ with using the similar way of the decomposition (Fig. 29). The same is true for extensions of diagonal U-split-complex numbers, etc. Norms and the scalar product of vectors can be defned in multi-dimensional spaces, which correspond to such systems of diagonal U-complex numbers (Petoukhov, 2017, version 8).

The limited volume of this article does not allow descripting the following interrelated data, which will be published in separate articles:

- Other kinds of decompositions of the Walsh-representation W_3, which lead to other systems of 4-block U-complex numbers, 4-block U-split-complex numbers, 2-block U-quaternions (non-Hamiltonian quaternions);
- Algebraic properties of these U-hypercomplex numbers concerning permutation transformations in triplets and their matrix $[C,A;T,G]^{(3)}$;
- Appropriate U-numerical sub-alphabets, which arise in these cases;
- Generalizations of the considered U-numerical systems to systems of 2^n-block U-hypercomplex numbers of different types;
- Applications of U-hypercomplex numbers in developing "personalized algebra-logical genetics", models of cyclic and oncological processes, biomechanics of human and animal gait, creation of systems of artificial intelligence, etc.

8. Some concluding remarks

On the basis of the matrix-algebraic study of genetic code systems we reveal and propose a new mathematical tool in a form of multi-block U-hypercomplex numbers. This tool seems to be useful for mathematical modeling of different structures. The vastness of its applications will be determined in large part by those researchers who want to apply it for different tasks. The introduced systems of multi-block U-numbers define new – for mathematical natural sciences – types of multi-dimensional spaces (U-numerical spaces), some of which are vector spaces with special interrelations of their vectors.

After the discovery of non-Euclidean geometries and of Hamilton quaternions, it is known that different natural systems can possess their own geometry and their own algebra (see about this (Kline, 1980)). But what kinds of algebra are appropriate for living organisms in their bio-informational aspects? We assume that inherited bio-informational aspects of organization of living matter is governed by systems of 2^n-block U-hypercomplex numbers. It seems that many difficulties of modern bioinformatics and mathematical biology are connected with utilizing – for biological structures – inadequate algebras, which were developed for completely other natural systems. Hamilton had similar difficulties in his attempts to describe transformations in a three-dimensional space by means of 3-dimensional numbers while this description needs 4-parametrical quaternions.

The proposed systems of 2^n-block united-hypercomplex numbers (U-hypercomplex numbers) generalize systems of complex numbers and hypercomplex numbers, which become a particular case of appropriate systems of multi-block U-hypercomplex numbers when only one of their blocks differs from zero. Hypercomplex numbers, beginning from quaternions of Hamilton, were created as an extension of complex numbers by means of including new independent imaginary units in addition to a single imaginary unit of complex numbers. In our studies of genetic matrices we paid attention on the alternative way of extentions of complex numbers: such extension is provided by means of including new independent real units (one or more real units, which we call local-real units) together with their particular complect of imaginary units (local-imaginary units). As far as we know, the proposed systems of united-hypercomplex numbers have not yet been encountered in mathematical natural science, and therefore they are the basis of a new class of mathematical models, primarily in biology. In mathematical natural science and computer science, single-block hypercomplex numbers are traditionally used; our genetic-mathematical researches draw attention to multi-block systems of hypercomplex numbers and corresponding matrix operators that allow working with multidimensional spaces, different subspaces of which are governed by different systems of hypercomplex numbers.

The modern mathematical natural sciences and technologies, a progress in which changes life of people, are based in an essential degree on using multidimensional numbers, which are called in a general case as hypercomplex numbers. Our described results in the field of genetic-mathematical researches have led to their generalization in a matrix form of united-hypercomplex numbers. It is natural to think that such generalization of the numerical basis can lead to useful extensions in different sciences and technologies with arising of appropriate extended theories and new practical applications in them. As known, matrices have been introduced in theoretical physics due to the creation of matrix mechanics of W. Heisenberg. From that time matrices became one of the most important tools in mathematical natural sciences and informatics. U-hypercomplex numbers in their matrix representations further strengthen the value of matrices in science.

In the author's opinion, the mathematics of U-hypercomplex numbers is beautiful and it can be used for further developing of algebraic biology, theoretical physics and informatics in accordance with the famous statement by P. Dirac, who taught that a creation of a physical theory must begin with the beautiful mathematical theory: "*If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics*" (this quotation is taken from (Arnold, 2006)). According to Dirac, all new physics, including relativistic and quantum, are developing in this way.

Materials of this article reinforce the author's point of view that living matter in its informational fundamentals is organized on algebraic bases. The author believes that a development of algebraic biology, elements of which are contained in this and other author's articles, is possible. By analogy with the known fact that molecular foundations of molecular genetics turned up unexpectedly very simple, perhaps algebraic foundations of living matter are also relative simple. Materials of our article give additional data to the themes about parallelisms in functioning of living organisms and computers and about special features of reading of genetic information (Carlevaro et al., 2016a,b; Igamberdiev and Shklovskiy-Kordi, 2016; Nemzer, 2017; Rapoport, 2016; Seligmann, 2011; Shu, 2017; and others). These materials are useful, in particular, for developing genetic biomechanics, biomechanics of man-machine systems, artificial intelligence, etc.

The American journal, Time, in 2008, announced "personalized genetics" from the company, 23andMe, as the best innovation of the year (Hamilton, 2008). This innovation was recognized to be much more important than many others, including the Large Hadron Collider from the field of nuclear physics. The company, 23andMe, proposes information about genetic peculiarities of persons at a low price. Now possibilities of personalized genetics connected with personal pharmacology are developed intensively in many countries with huge financial supports. But this initial kind of "personalized genetics" uses knowledge about the genetic code of protein sequences of amino acids without knowledge about the geno-logical coding (Petoukhov, 2016b; Petoukhov and Petukhova, 2017a,b). It is natural to think that the cause of the body's genetic predisposition to various diseases is not only violations in amino acid sequences of proteins, but that geno-logical disorders in inheritance of various processes also play an important role. Taking into account his concept of geno-logical coding and the described data about relations of genetic systems with U-hypercomplex numbers, the author proposes to develop "personalized algebra-logical genetics" with using the knowledge about U-hypercomplex numbers and about a set of U-numerical sub-alphabets, only some of which are represented in this article in (Figs. 15, 16, 19, 21).

Multi-block structure, which is genetically inherited, is one of the most characteristic features of a plurality of living bodies on different levels and branches of biological evolution. We believe that systems of multiblock U-hypercomplex numbers represent a relevant mathematics for modeling such inherited structures. But

$$\begin{vmatrix} C, A \\ T, G \end{vmatrix} \otimes \begin{vmatrix} C, A \\ T, G \end{vmatrix} = \begin{vmatrix} C\begin{vmatrix} C, A \\ T, G \end{vmatrix}, & A\begin{vmatrix} C, A \\ T, G \end{vmatrix} \\ T\begin{vmatrix} C, A \\ T, G \end{vmatrix}, & G\begin{vmatrix} C, A \\ T, G \end{vmatrix} \end{vmatrix} = \begin{vmatrix} CC, CA, AC, AA \\ CT, CG, AT, AA \\ TC, TA, GC, GG \\ TT, TG, GT, GG \end{vmatrix}$$

Fig. 31. The construction of the second tensor power of the alphabetic matrix [C, A; T, G] (see Fig. 3).

non-living substances also consist of various blocks of different levels up to atoms and elementary particles (one of the most important statements in the history of science has been done by Democritus approximately in the fourth century BC: "All bodies are composed of discrete units – atoms"). It is not excluded that the proposed systems of multi-block U-hypercomplex numbers will be also useful in theoretical physics to study block structures in non-living matter.

The notion of "number" is the main notion of mathematics and mathematical natural sciences. "*Complexity of a civilization is reflected in complexity of numbers used by this civilization*" (Davis, 1964). "*Number is one of the most fundamental concepts not only in mathematics, but also in all natural sciences. Perhaps, it is the more primary concept than such global categories, as time, space, substance or a field.*" (Pavlov, 2004).

Pythagoras studied figurate numbers and has formulated the idea: "*all things in the world are numbers*" or "*number rules the world*". B. Russell noted that he did not know of any other man who has been as influential as Pythagoras was in the sphere of thought (Russel, 1967). From this standpoint, there is no more fundamental scientific idea in the world, than this idea about a basic meaning of numbers. As Heisenberg noted, modern physics is moving along the same path along which the Pythagoreans walked (Heisenberg, 2000). Our genetic-mathematical researches have led to new systems of multidimensional numbers and have given new materials to the great idea by Pythagoras since these materials allow putting forward the hypothesis that genetic-informational organization of living matter are governed by multidimensional U-numbers (conditionally and briefly speaking – in the manner of the Pythagoras' slogan – "U-numbers rule living matter"). Initial researches in this direction should be prolonged.

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Appendix A. About tensor multiplication of matrices

The tensor (or Kronecker) multiplication of matrices, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix (Bellman, 1960). If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is a $p \times q$ matrix, then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the $mp \times nq$ block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

Fig. 31 shows the example how the second tensor power of the alphabetic matrix [C, A; T, G], which is represented in Fig. 3, is constructed.

Tensor products of matrices possess many interesting properties (Bellman, 1960).

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