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COMPLEX AND HYPERBOLIC FIBONACCI NUMBERS AND PHYLLOTAXIS

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Abstract: *The article is devoted to studying the extensions of the additive Fibonacci sequence into the additive sequences of 2-dimensional complex and hyperbolic numbers having Fibonacci coordinates. This study is connected with the active use of complex and hyperbolic numbers by contemporary sciences. Special attention is paid to features of the matrix representations of complex and hyperbolic Fibonacci numbers. The authors believe that complex and hyperbolic Fibonacci numbers, having interesting features, will be useful in different scientific fields.*

Keywords: Fibonacci numbers, phyllotaxis laws, complex numbers, hyperbolic numbers, matrix representation, eigenvalues.

1 INTRODUCTION

Various fields of mathematics, computer technologies and informatics, economics, and biology actively use Fibonacci numbers and their applications. Applications of Fibonacci numbers include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. A few examples of applications of Fibonacci numbers are the following.

The Fibonacci numbers can be defined by a Diophantine equation, which led to solving Hilbert's tenth problem [Harizanov, 1995]. There are known computers based on Fibonacci numbers [Stakhov, 2009]. A generalized Fibonacci sequence can be connected to the field of economics including the Brock–Mirman economic growth model [Brasch, Byström, Lystad, 2012]. The Fibonacci numbers are also an example of a complete sequence: this means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most. Fibonacci numbers are used by some pseudorandom number generators [Barker, 2012]. They are also used in planning poker, which is a step in estimating in software development projects that use the scrum methodology [Cohn, 2005]. The Fibonacci cube [Klavžar, 2011] is an undirected graph with a Fibonacci number of nodes that has been proposed as a network topology for parallel computing. Fibonacci retracement levels are widely used in technical analysis for financial market trading [Aspray, 2011]. The Fibonacci numbers are realized in biological phenomena of phyllotaxis at different levels and branches of biological evolution [Darvas, 2007; Jean, 2006; Petoukhov, 2008; Petoukhov, He, 2010]. They are closely connected with the golden ratio $\varphi = (1+5^{0.5})/2 = 1{,}618\ldots$, which is known in the aesthetics of proportions and many biological systems, as described in publications by many authors (see reviews in [Darvas, 2007; Darvas, et al. 2012; Olsen, 2006; Stakhov, 2009; Petoukhov, 2008; Petoukhov, He, 2010]). The golden ratio is also considered in works about Klein bottle logophysics and evolution (see, for example, [Rapoport, 2016; Rapoport, Perez, 2018]).

The Fibonacci series is an additive sequence of real numbers F_n satisfying the equation $F_{n+2} = F_n + F_{n+1}$, $F_1 = F_2 = 1$ (Table 1).

n	1	2	3	4	5	6	7	8	9	10	…
F_n	1	1	2	3	5	8	13	21	34	55	…

Table 1: The additive sequence of real Fibonacci numbers.

The ratios of two consecutive Fibonacci numbers tend to the golden ratio φ as *n* increases:

$$F_{n+1}/F_n: 2/1, 3/2, 5/3, 8/5, 13/8, 21/13, 34/21, \ldots \rightarrow \varphi \quad (1)$$

The main purpose of this article is to analyse and describe the properties of the additive series of complex and hyperbolic Fibonacci numbers, which are analogous to the real Fibonacci series. The interest in considering these series of 2-dimensional numbers, whose real and imaginary parts are real Fibonacci numbers, is due to the important role of multidimensional numbers in modern science.

Various kinds of multi-dimensional numbers – complex numbers, hyperbolic numbers, dual numbers, quaternions and other hypercomplex numbers – are used in different branches of modern science. They have played the role of the magic tool for the development of theories and calculations in problems of heat, light, sounds, fluctuations, elasticity, gravitation, magnetism, electricity, current of liquids, quantum-mechanical phenomena, special theory of relativity, nuclear physics, etc.

Among many types of multidimensional numbers, 2-dimensional complex and hyperbolic numbers occupy an important place [Kantor, Solodovnikov, 1989]. Their special additive Fibonacci-type series are considered below - in connection with the phyllotaxis laws – as analogues of the series of real Fibonacci numbers. There were the parastichy sequences of phyllotaxis laws, consisting of ratios of pairs of neighbouring numbers of the Fibonacci series, that provoked the authors to the construction and analysis of complex and hyperbolic Fibonacci numbers and their additive series. The authors believe that the proposed additive sequences of two-dimensional Fibonacci numbers, also related to the golden ratio by a similar algorithm, have prospects for applications in different areas, like the classical series of real Fibonacci numbers.

2 COMPLEX FIBONACCI NUMBERS AND PHYLLOTAXIS

Complex numbers have extremely wide applications in many scientific areas, including signal processing, control theory, electromagnetism, fluid dynamics, cartography, vibration analysis, etc. Note especially that the complex number field is intrinsic to the mathematical formulations of quantum mechanics; the original foundation formulas of quantum mechanics – the Schrödinger equation and Heisenberg's matrix mechanics – make use of complex numbers.

The following citation characterizes the important meaning of complex numbers: "Complex numbers, as much as reals, and perhaps even more, find a unity with nature that is truly remarkable. It is as though Nature herself is as impressed by the scope and consistency of the complex-number system as we are ourselves, and has entrusted to these numbers the precise operations of her world at its minutest scales" [Penrose, 2016, p. 73].

Every complex number can be expressed in the form $z = x + yi$, where x and y are real numbers and element i, called the imaginary unit, satisfies the equation $i^2 = -1$. The complex numbers form a rich structure that is simultaneously an algebraically closed field, a commutative algebra over the reals, and a Euclidean vector space of dimension two.

A complex number $x+yi$ has its matrix form of representation: [x, -y; y, x] = x*[1, 0; 0, 1] +y*[0, -1; 1, 0] where [1, 0; 0, 1] is the identity matrix representing real basic unit; the matrix [0, -1; 1, 0] represents imaginary basic unit i. If $x^2+y^2 = 1$, then the matrix [x, -y; y, x] defines rotations.

In biology, it has long been known that, for example, the spiral arrangement of many plant objects' bioorganisms form ordered patterns (shoots of plants and trees, seeds in the heads of sunflowers, scales of coniferous cones and pineapples, etc.) [Jean, 2006]. These patterns are determined by the overlapping left and right-oriented spiral lines – parastichies. The phyllotaxis of such botanical objects usually indicates two parameters: the number of left spirals and the number of right spirals, which are observed on the surface of phyllotaxis objects. Phyllotaxis structures with such patterns are described by the sequence of ratios of two neighbouring Fibonacci numbers (1), which is traditionally called the parastichic sequence [Jean, 2006; Petoukhov, 1981].

Taking into account the correspondence of the phyllotaxis to the sequence (1), the members of which consist of the union of two neighbouring numbers of the Fibonacci series into a single object, and also the fact that the spirals of the phyllotaxis are conjugated with transformations of turns, it seems natural to turn to the complex numbers $z = x + iy$, each of which also consists of two real numbers x and y combined into a single object and whose matrix representations are associated with operators of turns. In this regard, we represent each ratio F_{n+1}/F_n in the series (1) by the complex number $P_n = F_{n+1} + iF_n$ that we call the complex Fibonacci number of the parastichic type (Table 2).

n	1	2	3	4	5	6	7	8	…
P_n	1+i	2+i	3+2i	5+3i	8+5i	13+8i	21+13i	34+21i	…

Table 2: The sequence of complex Fibonacci numbers P_n of the parastichic type.

This action transforms the additive parastichy Fibonacci sequence (1) into an additive parastichy sequence of complex Fibonacci numbers P_n (2), which can also be represented as an additive parastichy sequence of matrix representations of these numbers P_n (3).

$$P_n = F_{n+1}+iF_n: 1+i,\ 2+i,\ 3+2i,\ 5+3i,\ 8+5i,\ 13+8i,\ 21+13i,\ \ldots \quad (2)$$

$$\begin{vmatrix} F_{n+1}, & -F_n \\ F_n, & F_{n+1} \end{vmatrix} : \begin{vmatrix} 1, & -1 \\ 1, & 1 \end{vmatrix} ; \begin{vmatrix} 2, & -1 \\ 1, & 2 \end{vmatrix} ; \begin{vmatrix} 3, & -2 \\ 2, & 3 \end{vmatrix} ; \begin{vmatrix} 5, & -3 \\ 3, & 5 \end{vmatrix} ; \begin{vmatrix} 8, & -5 \\ 5, & 8 \end{vmatrix} ; \ldots \quad (3)$$

Complex Fibonacci numbers P_n (2) and their matrix representations (3) have interesting properties, which include the following. The series of ratios of two consecutive complex Fibonacci numbers P_{n+1}/P_n tends to the golden ratio φ as n increases (by analogy with expression (1) for real Fibonacci numbers). Table 3 shows the values of initial members of the series of ratios P_{n+1}/P_n.

n	1	2	3	4	…
P_{n+1}/P_n	1.50-0.5i	1.60+0.2i	1.615-0.08i	1.618+0.03i	…

Table 3: The values of the ratios P_{n+1}/P_n of the first neighbouring members of the series of complex Fibonacci numbers.

Let us prove that the sequence of ratios of complex Fibonacci numbers of parastichic type P_{n+1}/P_n converges to the golden section. In this proof, we will use the known properties of the real Fibonacci numbers, presented by equalities (4) and (5) (taken from [Krzywkowski, 2010]).

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \quad (4)$$

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \quad (5)$$

We express the ratio P_{n+1}/P_n in terms of the corresponding real Fibonacci numbers using (4) and (5), as well as the simultaneous multiplication of the numerator and denominator of this ratio by a complex number (F_n- iF_{n+1}):

$$P_{n+1}/P_n = (F_{n+1}+iF_n) / (F_n+iF_{n+1}) =$$

$$= \{(F_{n+1}+iF_n)(F_n- iF_{n+1})\} / \{(F_n+ iF_{n+1})(F_n- iF_{n+1})\} =$$

$$= \{F_n(F_{n+1}+F_{n-1}) - \mathrm{i}(F_{n+1}F_{n-1} -F_n^2)\}/ (F_n^2+F_{n-1}^2) =$$

$$= F_n(F_{n+1}+F_{n-1}) / F_{2(n-1)+1} - i(F_{n+1}F_{n-1} -F_n^2) / F_{2(n-1)+1} =$$

$$= F_{2n}/F_{2n-1} - i(-1)^n/F_{2n-1} \qquad (6)$$

The resulting expression (6) shows that as n increases, the real part F_{2n}/F_{2n-1} of the ratio P_{n+1}/P_n tends to the golden ratio φ following (1), and the imaginary part $(-1)^n/F_{2n-1}$ tends to zero. This result confirms that the parastichy sequence of complex Fibonacci numbers (2) is an extension of the parastichy sequence of real Fibonacci numbers (1).

Now, let us turn to the squared norm $||P_n||$ of a complex Fibonacci number $P_n = F_{n+1}+ iF_n$. As it is known, the squared norm $||z||$ (or absolute square, or the vector norm) of a complex number $z = x+iy$ is equal to its product with its conjugate number $x-iy$ [https://mathworld.wolfram.com/AbsoluteSquare.html]. Correspondingly, you have the expression (7) for the squared norm $||P_n||$ of the complex Fibonacci number P_n:

$$||P_n|| = (F_{n+1}+ iF_n)(F_{n+1}-iF_n) = F_{n+1}^2 + F_n^2 \qquad (7)$$

Taking into account the equations (4) and (7), you get Table 4 showing the series of squared norms of complex Fibonacci numbers P_n.

n	1	2	3	4	5	6	7	8	…
$\|\|P_n\|\| = F_{2n+1}$	2	5	13	34	89	233	610	1597	…

Table 4: The series of squared norms $||P_n||$ of complex Fibonacci numbers from Table 2.

The sequence of squared norm ratios $||P_{n+1}|| / ||P_n||$ of neighbouring complex Fibonacci numbers has the form (8) and is a sparse version of the well-known phyllotaxis orthostich sequence converging to the square of the golden section $φ^2$ [Jean, 2006]:

$||P_{n+1}||/||P_n||$: 5/2, 13/5, 34/13, 89/34, 233/89, …→ φ^2 = 2,618… (8)

The value of the argument α of the complex Fibonacci number $P_n = F_{n+1} + iF_n$ in radians is given by the expression $\alpha = \text{arctg}\,(F_n/F_{n+1})$.

Each of the matrices $[F_{n+1}, -F_n; F_n, F_{n+1}]$, representing a complex Fibonacci number P_n, has two eigenvalues $F_{n+1} + iF_n$ and $F_{n+1} - iF_n$, which are two conjugate complex Fibonacci numbers.

The matrix form of the hyperbolic Fibonacci numbers is a matrix operator that symmetrically transforms any figure (as a system of points) on the hyperbolic plane into a new figure.

One can believe that using complex Fibonacci numbers gives new possibilities for applications of Fibonacci numbers in different fields, including the study of fractals and bio-symmetry (initially fractal patterns were created by Mandelbrot on the base of complex numbers). The authors think that complex Fibonacci numbers are also useful for some applications in quantum biology, which uses complex numbers and which is under development by many authors.

3 HYPERBOLIC FIBONACCI NUMBERS AND PHYLLOTAXIS

Hyperbolic numbers are also used in different scientific fields including relativistic kinematics, biomechanics, and bio-symmetrology [Bodnar, 1992, 1994; Hu, Petoukhov, 2017; Petoukhov, 2008, 2011, 2016; Smolyaninov, 2000]. In abstract algebra, a hyperbolic number $g = x + yj$ (or split-complex number, also double number, perplex number) has two real number components x and y, and the imaginary basic unit $j \neq \pm 1$ but $j^2 = +1$ [Kantor, Solodovnikov]. All hyperbolic numbers form an algebra over the field of real numbers and are located on the hyperbolic plane. This algebra is not a division algebra or field since it contains zero divisors. Addition and multiplication of hyperbolic numbers are defined by (9):

$$(x+jy)+(u+jv)=(x+u)+j(y+v); \quad (x+jy)(u+jv)=(xu+yv)+j(xv+yu) \tag{9}$$

This multiplication is commutative, associative and distributed over addition.

A hyperbolic number has its symmetric matrix form of representation: $[x, y; y, x] = x*[1, 0; 0, 1] + y*[0, 1; 1, 0]$ where [1, 0; 0, 1] is the identity matrix representing real basic unit; the matrix [0, 1; 1, 0] represents imaginary basic unit. If x^2-y^2 = 1, then the matrix [x, y; y, x] defines hyperbolic rotations known in the special theory of relativity as a Lorentz transformation. Hyperbolic rotations are usually expressed by a symmetric matrix [coshα, sinhα; sinhα, coshα] through hyperbolic cosine «cosh» and hyperbolic sine «sinh».

Symmetric matrices that represent hyperbolic numbers have real eigenvalues and orthogonal eigenvectors (which distinguishes them from non-symmetric matrix representations of complex numbers). Such symmetric matrices form the basis of the theory of resonances of oscillatory systems with many degrees of freedom and are also metric tensors from the point of view of Riemannian geometry.

Taking into account that, in the ontogenetic processes of plant growth, phyllotaxis patterns are transformed by hyperbolic rotations [Bodnar, 1992, 1994], it seems natural to use 2-dimensional hyperbolic numbers $g = x + yj$ for modelling members of the parastichic series (1). In the proposed approach, the parastichy sequence (1) of ratios is transformed into additive sequences (10), (11) reflecting linear notation of appropriate hyperbolic numbers and their matrix presentations (we term sequences (10), (11) as parastichy sequences of hyperbolic numbers):

$$F_{n+1} + jF_n\text{: } 2 + j,\ 3 + j2,\ 5 + 3j,\ 8 + 5j,\ 13 + 8j,\ 21 + 13j,\ \ldots \qquad (10)$$

$$\left|\begin{matrix} F_{n+1}, F_n \\ F_n,\ F_{n+1} \end{matrix}\right| : \left|\begin{matrix} 2, 1 \\ 1, 2 \end{matrix}\right|, \left|\begin{matrix} 3, 2 \\ 2, 3 \end{matrix}\right|, \left|\begin{matrix} 5, 3 \\ 3, 5 \end{matrix}\right|, \left|\begin{matrix} 8, 5 \\ 5, 8 \end{matrix}\right|, \left|\begin{matrix} 13, 8 \\ 8, 13 \end{matrix}\right| \ldots \qquad (11)$$

Now let us describe the results of the study of eigenvalues of the symmetric matrices in the parastichic sequence (11). Each of these matrices [F_{n+1}, F_n; F_n, F_{n+1}] has two eigenvalues, which are equal to two Fibonacci numbers again: F_{n+2} and F_{n-1}. One can noted that these eigenvalues are the sum and the difference of the Fibonacci components of the original hyperbolic number F_{n+1}+ jF_n since $F_{n+2} = F_{n+1}$+ F_n and $F_{n-1} = F_{n+1} - F_n$. The ratio F_{n+2}/F_{n-1} of such eigenvalues defines a new sequence (12) of Fibonacci ratios, which tend to φ^3 as n increases:

$$F_{n+2}/F_{n-1}\text{:} \quad 3/1,\ 5/1,\ 8/2,\ 13/3,\ 21/5,\ 34/8,\ 55/13,\ \ldots \rightarrow \varphi^3 \qquad (12)$$

Such a pair of eigenvalues F_{n+2} and F_{n-1} can be considered as components of a new hyperbolic number $F_{n+2} + jF_{n-1}$. In this case, the sequence of ratios (12) is transformed into additive sequences (13) and (14) reflecting linear notations of appropriate hyperbolic numbers and their matrix presentations:

$$F_{n+2}+jF_{n-1}: \quad 3+j,\ 5+j,\ 8+j2,\ 13+j3,\ 21+j5,\ 34+j8,\ \ldots. \tag{13}$$

$$\begin{vmatrix} F_{n+2}, F_{n-1} \\ F_{n-1}, F_{n+2} \end{vmatrix} : \begin{vmatrix} 3, 1 \\ 1, 3 \end{vmatrix}, \begin{vmatrix} 5, 1 \\ 1, 5 \end{vmatrix}, \begin{vmatrix} 8, 2 \\ 2, 8 \end{vmatrix}, \begin{vmatrix} 13, 3 \\ 3, 13 \end{vmatrix}, \ldots \tag{14}$$

Each of the symmetric matrices $[F_{n+2}, F_{n-1}; F_{n-1}, F_{n+2}]$ of the sequence (14) has two eigenvalues, which are again equal to two Fibonacci numbers multiplied by a factor 2 (twice the Fibonacci numbers): $2F_{n+1}$ and $2F_n$. Ratios $2F_{n+1}/2F_n$ of such eigenvalues form a sequence, which is identical to the initial parastichy sequence (1).

This procedure of analysis of the eigenvalues of new and new sequences of symmetric matrices, representing hyperbolic numbers by analogy with sequences (10)-(14), can be repeated as long as desired, obtaining a hierarchy of eigenvalues of the matrices based on Fibonacci numbers multiplied by a factor 2 at corresponding steps of the iterative procedure.

In many ways, similar results are obtained by considering the additive series of two-dimensional hyperbolic Lucas numbers and the additive series of their matrix representations, which determine the additive series of eigenvalues of these symmetric matrices. Here one can remind that one-dimensional Lucas numbers form the series $L_{n+2} = L_n + L_{n+1}$: 2, 1, 3, 4, 7, 11, 18, ..., which is also known in phyllotaxis laws [Jean, 2006].

It should be noted that the study of the eigenvalues of symmetric matrices has special meaning since, in the theory of oscillations, symmetric matrices are matrix representations of oscillatory systems with many degrees of freedom. Moreover, the eigenvalues of such a matrix determine the resonant frequencies of the corresponding oscillatory system. The described results on the properties of inherited phyllotaxis phenomena with their Fibonacci ratios, represented by symmetric matrices and their matrix eigenvalues, are important, in particular, for the concept of multi-resonance genetics, which connects structural features of molecular-genetic systems with resonances of oscillatory systems [Petoukhov, 2016].

The matrix form of the hyperbolic Fibonacci numbers is a matrix operator that symmetrically transforms on the hyperbolic plane any figure (as a system of points) into a new figure.

4 SOME CONCLUDING REMARKS

The development of modern mathematical natural sciences is based on the use of certain mathematical tools. Mathematical tools of theoretical research can be compared with glasses for a visually impaired person: adequate glasses provide a person with a clear and beautiful picture of reality, which he had previously seen as blurred and hidden by fog. This article attracts the attention of researchers to interesting features of additive sequences of 2-dimensional complex and hyperbolic numbers, having Fibonacci coordinates. In particular, these multidimensional numbers can be used for modelling some biological symmetries including phyllotaxis phenomena.

In contrast to the traditional study of the series of Fibonacci numbers in the framework of one-dimensional real numbers, the study of complex and hyperbolic numbers with Fibonacci coordinates proposed by the authors additionally introduces important mathematical objects, for example, eigenvalues and eigenvectors of matrix representations of the named multidimensional numbers.

In particular, the matrix form of presentation of hyperbolic numbers deserves special attention for the following reasons:

1) This presentation form is based on symmetric matrices, which are closely related to the theory of resonances of oscillatory systems, having many degrees of freedom, and also with Punnett squares from Mendelian genetics of inheritance of traits in living organisms [Petoukhov, 2011, 2016];

2) These symmetric matrices can be interpreted as metric tensors, which are the main invariants in the Riemannian geometry and which can be used in the theory of morpho-resonance morphogenesis [Petoukhov, 2008, 2016; Petoukhov, He, 2010];

3) These symmetric matrices are related to hyperbolic rotations, which are particular cases of hyperbolic numbers and are connected with the theory of biological phyllotaxis laws, problems of locomotion control [Smolyaninov, 2010], and also with Lorenz transformations in the special theory of relativity;

4) These symmetric matrices are related to the theory of solitons of the sine-Gordon equation. Such known solitons are the only relativistic type of solitons; they were put forward for the role of the fundamental type of solitons of living matter in the book [Petoukhov, 1999].

The proposed approach connects real Fibonacci numbers with multi-dimensional complex and hyperbolic numbers. It expands modern possibilities of constructing new applications and theories based on Fibonacci numbers and matrix representations of multidimensional numbers. One of the interesting directions is the application of complex and hyperbolic numbers with Fibonacci coordinates in generalized crystallography, genetic biomechanics, and biochemical aesthetics [Petoukhov, 2008; Hu, Petoukhov, 2017].

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