Symmetry: Culture and Science
Vol. 23, No.3-4, 275-301, 2012

SYMMETRIES OF THE GENETIC CODE, HYPERCOMPLEX NUMBERS AND GENETIC MATRICES WITH INTERNAL COMPLEMENTARITIES

S.V.Petoukhov

biophysics, bioinformatics (b. Moscow, Russia, 1946).

Address: Laboratory of Biomechanical Systems, Mechanical Engineering Research Institute of Russian Academy of Sciences; Malyi Kharitonievskiy pereulok, 4, Moscow, 101990, Russia. E-mail: spetoukhov@gmail.com.

Fields of interest: genetics, bioinformatics, biosymmetries, multidimensional numbers, musical harmony, mathematical crystallography (also history of sciences, oriental medicine).

Awards: Gold medal of the Exhibition of Economic Achievements of the USSR, 1974; State Prize of the USSR, 1986; Honorary diplomas of a few international conferences and organizations, 2005-2012.

Publications and/or Exhibitions: 1) S.V. Petoukhov (1981) *Biomechanics, Bionics and Symmetry*. Moscow, Nauka, 239 pp. (in Russian); 2) S.V. Petoukhov (1999) *Biosolitons. Fundamentals of Soliton Biology*. Moscow, GPKT, 288 pp. (in Russian); 3) S.V. Petoukhov (2008) *Matrix Genetics, Algebras of the Genetic Code, Noise-immunity*. Moscow, RCD, 316 pp. (in Russian); 4) S.V. Petoukhov, M. He (2010) *Symmetrical Analysis Techniques for Genetic Systems and Bioinformatics: Advanced Patterns and Applications*, Hershey, USA: IGI Global, 271 pp.; 5) He M., Petoukhov S.V. (2011) Mathematics of Bioinformatics: Theory, Practice, and Applications. USA: John Wiley & Sons, Inc., 295 pp.

Abstract: *The article describes results of study of some symmetries of the genetic coding system by means of matrix representations of its molecular ensembles. This matrix approach is borrowed by the author from the known theory of noise-immunity coding, which is used for a long time in discrete signals processing for communication and computer technology. In the process, important connections between the hierarchy of genetic alphabets and complex numbers, quaternions by Hamilton and some other multi-dimensional numbers are discovered by means of analysis of reasoned numeric representations of genetic* (2^n*2^n)*-matrices. It has been shown that these numeric matrices belong to a class of "matrices with internal complementarities" and they allow creation of new mathematical tools to study the molecular-genetic system,* *including hidden regularities of long nucleotide sequences. The described results give some evidences about the algebraic nature of the molecular-genetic system.*

Keywords: symmetry, genetic code, matrix, hypercomplex numbers, complementarity, Kronecker multiplication, long nucleotide sequences.

1. ABOUT THE PARTNERSHIP OF THE GENETIC CODE AND MATHEMATICS

Science has led to a new understanding of life itself: "*Life is a partnership between genes and mathematics*" (Stewart, 1999). This article describes a system of multidimensional numeric structures together with some evidences that this mathematical system is the partner of molecular ensembles of the genetic code. The described results are based on symmetric properties of the genetic code system and on a matrix approach which was borrowed by the author from mathematics of noise-immunity coding to study genetic phenomenology (Petoukhov, 2008a-c, 2011, 2012; Petoukhov, He, 2010).

H_4 =

1	1	-1	1
-1	1	1	1
1	-1	1	1
-1	-1	-1	1

; **H_8** =

1	-1	1	-1	-1	1	1	-1
1	1	1	1	-1	-1	1	1
-1	1	1	-1	1	-1	1	-1
-1	-1	1	1	1	1	1	1
1	-1	-1	1	1	-1	1	-1
1	1	-1	-1	1	1	1	1
-1	1	-1	1	-1	1	1	-1
-1	-1	-1	-1	-1	-1	1	1

R_4 =

1	1	1	-1
-1	1	-1	-1
1	-1	1	1
-1	-1	-1	1

; **R_8** =

1	1	1	1	1	1	-1	-1
1	1	1	1	1	1	-1	-1
-1	-1	1	1	-1	-1	-1	-1
-1	-1	1	1	-1	-1	-1	-1
1	1	-1	-1	1	1	1	1
1	1	-1	-1	1	1	1	1
-1	-1	-1	-1	-1	-1	1	1
-1	-1	-1	-1	-1	-1	1	1

Figure 1: numeric matrices H_4, H_8, R_4 and R_8 which are connected with phenomenology of the genetic coding system (Petoukhov, 2011, 2012)

The main mathematical objects of the article are four matrices R_4, R_8, H_4 and H_8 shown on Figure 1. Why these numeric matrices are chosen from infinite set of matrices? The reason is that they are connected with phenomenology of the genetic code system in matrix forms of its representation as it was shown in works (Petoukhov, 2011, 2012), and as it will be additionally demonstrated in the end of this article, where a conclusion about algebraic essence of the nature of genetic informatics will be made. The matrices H_4 and H_8 belong to a huge set of famous Hadamard matrices, which are widely used for noise-immunity coding in technologies of signals processing. The matrices R_4 and R_8 are conditionally termed "Rademacher matrices" because each of their columns represents one of known Rademacher functions.

2. THE HADAMARD MATRICES H_4 AND H_8

Let us begin with analysis of the (4*4)-matrix H_4 (Figure 1). One of variants of decomposition of the matrix H_4 gives a set of 4 sparse matrices H_{40}, H_{41}, H_{42} and H_{43} (Figure 2). This set is closed in relation to multiplication and it defines their multiplication table (Figure 2, bottom row) that is identical to the famous multiplication table of quaternions by Hamilton. From this point of view, the matrix H_4 is the quaternion by Hamilton with unit coordinates. (Such type of decompositions is termed a dyadic-shift decomposition because it corresponds to structures of matrices of dyadic shifts, well known in technology of signals processing (Ahmed, Rao, 1975)).

$H_4 = H_{40} + H_{41} + H_{42} + H_{43} =$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} + \begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix} + \begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}$$

	1	H_{41}	H_{42}	H_{43}
1	1	H_{41}	H_{42}	H_{43}
H_{41}	H_{41}	- 1	H_{43}	- H_{42}
H_{42}	H_{42}	- H_{43}	- 1	H_{41}
H_{43}	H_{43}	H_{42}	- H_{41}	- 1

Figure 2: the dyadic-shift decomposition of the (4*4)-matrix H_4 (from Figure 1) gives the set of 4 sparse matrices H_{40}, H_{41}, H_{42} and H_{43}, which corresponds to the multiplication table of quatrnions by Hamilton (bottom row). The matrix H_{40} is identity matrix

But the matrix H_4 is also the sum of two sparse matrices HL_4 and HR_4 (Figure 3). One can numerate 4 columns of the matrix H4 from left to right by numbers 0, 1, 2 and 3. In this case two columns with non-zero entries in the matrix HL_4 have numerations with even numbers 0 and 2; two columns with non-zero entries in the matrix HR_4 have numerations with odd numbers 1 and 3. In view of this, such decomposition H_4=HL_4 +HR_4 can be conditionally termed as "the even-odd decomposition" (such type of decompositions will be used a few times in this article).

$H_4 = HL_4 + HR_4 =$

1	0	-1	0
-1	0	1	0
1	0	1	0
-1	0	-1	0

+

0	1	0	1
0	1	0	1
0	-1	0	1
0	-1	0	1

,

$HL_4 = HL_{40} + HL_{41} =$

1	0	0	0
-1	0	0	0
0	0	1	0
0	0	-1	0

+

0	0	-1	0
0	0	1	0
1	0	0	0
-1	0	0	0

,

$HR_4 = HR_{40} + HR_{41} =$

0	1	0	0
0	1	0	0
0	0	0	1
0	0	0	1

+

0	0	0	1
0	0	0	1
0	-1	0	0
0	-1	0	0

Figure 3: upper row: the representation of the matrix H_4 as sum of matrices HL_4 and HR_4. Other rows: representations of each of matrices HL_4 and HR_4 as sums of two matrices: HL_4=HL_{40}+HL_{41}, HR_4 =HR_{40}+HR_{41}

It is unexpected but the set of two (4*4)-matrices HL_{40} and HL_{41} is also closed in relation to multiplication and it defines their multiplication table (Figure 43), identical to the multiplication table of complex numbers (http://en.wikipedia.org/wiki/Complex_number). One can note that in the field of matrix analysis, complex numbers are usually represented by means of (2*2)-matrices [a, -b; b, a]. Let us consider now the set of (4*4)-matrices C_L = a_0*HL_{40}+a_2*HL_{41} which is the unusual representation of complex numbers (here a_0, a_2 are real numbers) (Figure 4). The classical identity matrix E=[1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1] is absent in the set of matrices C_L, each of which has zero determinant. Consequently the usual notion of the inverse matrix C_L^{-1} (as C_L*C_L^{-1}=E) can't be defined in relation to the classical identity matrix E in accordance with the famous theorem about inverse matrices for matrices with zero determinant (Bellman, 1960, Chapter 6, § 4). On the other hand, the set of matrices C_L has the matrix HL_{40}, which possesses all properties of identity matrix (or the real unit) for any member of this set (one can check that the matrix HL_{40} represents the real unit in this set). In the frame of the set of matrices C_L, where the matrix HL_{40} represents the real unity, one can define the special notion of inverse matrix C_L^{-1} for any non-zero matrix C_L in relation to the matrix HL_{40} on the base of equations: $C_L*C_L^{-1} = C_L^{-1}*C_L = HL_{40}$. From this point of view, the genetic (4*4)-matrix HL_4 is the complex number with unit coordinates (a_0=a_2=1). In the case of genetic matrices, we reveal that 4-dimensional spaces can contain 2-parametric subspaces, in which complex numbers exist in the form of (4*4)-matrices C_L.

	HL_{40}	HL_{41}
HL_{40}	HL_{40}	HL_{41}
HL_{41}	HL_{41}	$-HL_{40}$

;

$C_L = a_0*HL_{40}+a_2*HL_{41} =$

a_0	0	$-a_2$	0
$-a_0$	0	a_2	0
a_2	0	a_0	0
$-a_2$	0	$-a_0$	0

$C_L^{-1} = (a_0^2+a_2^2)^{-1} *$

a_0	0	a_2	0
$-a_0$	0	$-a_2$	0
$-a_2$	0	a_0	0
a_2	0	$-a_0$	0

Figure 4: the multiplication table of two (4*4)-matrices HL_{40} and HL_{41} (from Figure 3), which represent a set of two basic elements of complex numbers $C_L = a_0*HL_{40}+a_2*HL_{41}$, where a_0, a_2 are real numbers. In the frame of the set of 2-parametric matrices C_L, where the matrix HL_{40} represents the real unit, the matrix C_L^{-1} is the inverse matrix for C_L by definition on the base of the equation: $C_L*C_L^{-1} = HL_{40}$

A similar situation holds true for (4*4)-matrices $HR_4 = HR_{40} + HR_{41}$ (from Figure 3). The set of two matrices HR_{40} and HR_{41} is also closed in relation to multiplication; it gives the multiplication table (Figure 5) which is also identical to the multiplication table of complex numbers. The set of (4*4)-matrices $C_R = a_1*HR_{40}+a_3*HR_{41}$, where a_1, a_3 are real numbers, represents complex numbers in the (4*4)-matrix form (Figure 5). The matrix HR_{40} plays a role of the real unit in this set of matrices C_R. In the frame of matrices C_R, where HR_{40} represents the real unit, the matrix C_R^{-1} (Figure 5) is the inverse matrix for any non-zero matrix C_R by definition on the base of equations $C_R*C_R^{-1} = C_R^{-1}*C_R = HR_{40}$. The genetic matrix HR_4 is complex number with unit coordinates (a_1=a_3=1). Two sets of (4*4)-matrices C_L and C_R are quite different representations of complex numbers; for example, a sum C_L+C_R of members of these sets is not complex number.

	HR_{40}	HR_{41}
HR_{40}	HR_{40}	HR_{41}
HR_{41}	HR_{41}	$-HR_{40}$

; $C_R = a_1*HR_{40}+a_3*HR_{41} =$

0	a_1	0	a_3
0	a_1	0	a_3
0	$-a_3$	0	a_1
0	$-a_3$	0	a_1

$C_R^{-1} = (a_1^2+a_3^2)^{-1} *$

0	a_1	0	$-a_3$
0	a_1	0	$-a_3$
0	a_3	0	a_1
0	a_3	0	a_1

Figure 5: the multiplication table of two (4*4)-matrices HR_{40} and HR_{41} (from Figure 3), which represent a set of two basic elements of complex numbers $C_R = a_1*HR_{40}+a_3*HR_{41}$, where a_1, a_3 are real numbers. In the frame of the set of 2-parametric matrices C_R, where the matrix HR_{40} represents the real unit, the matrix C_R^{-1} is the inverse matrix for any non-zero matrix C_R by definition on the base of the equation: $C_R*C_R^{-1} = HR_{40}$

One should note that actions of the (4*4)-matrices HL_4 and HR_4 on 4-dimensional vectors in their planes $R_0(x_0, 0, x_2, 0)$ and $R_1(0, x_1, 0, x_3)$ rotate the vectors in different directions: clockwise and counterclockwise (Figure 6). The properties of these genetic matrices can be used in studying the famous problem of dissymmetry in biological organisms.

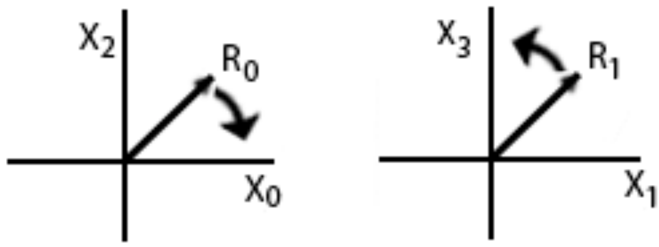

Figure 6: The action of the matrix HL_4 on a 4-dimensional vector $R_0(x_0, 0, x_2, 0)$ leads to a vector rotation clockwise (on the left). The action of the matrix HR_4 on a 4-dimensional vector $R_1(0, x_1, 0, x_3)$ leads to a vector rotation counterclockwise (on the right)

As described above, we have received one more interesting result: the sum of two 2-dimensional complex numbers HL_4 and HR_4 with unit coordinates (they belong to two different matrix types of complex numbers) generates the 4-dimensional quaternion by Hamilton with unit coordinates $H_4=HL_4+HR_4$ (Figure 2). It resembles a situation when a union of Yin and Yang (or a union of female and male beginnings, or a fusion of male and female gametes) generates a new organism. Below we will meet with other similar situations concerning (2^n*2^n)-matrices, which represent (2^n)-dimensional numbers with unit coordinates and which consists of two “complementary” halves (like the matrix H_4), each of which is 2^{n-1}-dimensional number with unit coordinates. One can name such type of matrices as “matrices with internal complementarities”. They resemble in some extend the complementary structure of double helixes of DNA.

Let us return now to the (8*8)-matrix H_8 (Figure 1) and demonstrate that it is also the matrix with internal complementarities. Figure 6 shows the matrix H_8 as sum of matrices HL_8 and HR_8.

$H_8 = HL_8 + HR_8 =$

1	0	1	0	-1	0	1	0		0	-1	0	-1	0	1	0	-1
1	0	1	0	-1	0	1	0		0	1	0	1	0	-1	0	1
-1	0	1	0	1	0	1	0		0	1	0	-1	0	-1	0	-1
-1	0	1	0	1	0	1	0	+	0	-1	0	1	0	1	0	1
1	0	-1	0	1	0	1	0		0	-1	0	1	0	-1	0	-1
1	0	-1	0	1	0	1	0		0	1	0	-1	0	1	0	1
-1	0	-1	0	-1	0	1	0		0	1	0	1	0	1	0	-1
-1	0	-1	0	-1	0	1	0		0	-1	0	-1	0	-1	0	1

Figure 7: The matrix H_8 (from Figure 1) is one of matrices with internal complementarities, which are represented by its halves HL_8 and HR_8 (explanation in text)

Figure 8 shows a decomposition of the matrix HL_8 (from Figure 7) as a sum of 4 matrices: $HL_8 = HL_{80} + HL_{81} + HL_{82} + HL_{83}$. The set of matrices HL_{80}, HL_{81}, HL_{82} and HL_{83} is closed in relation to multiplication and it defines the multiplication table which is identical to the multiplication table of quaternions by Hamilton. General expression for quaternions in this case can be written as $Q_L = a_0*HL_{80} + a_1*HL_{81} + a_2*HL_{82} + a_3*HL_{83}$, where a_0, a_1, a_2, a_3 are real numbers. From this point of view, the (8*8)-genomatrix HL_8 is the 4-dimensional quaternion by Hamilton with unit coordinates.

$HL_8 = HL_{80} + HL_{81} + HL_{82} + HL_{83} =$

$$\begin{pmatrix} 1&0&1&0&-1&0&1&0\\ 1&0&1&0&-1&0&1&0\\ -1&0&1&0&1&0&1&0\\ -1&0&1&0&1&0&1&0\\ 1&0&-1&0&1&0&1&0\\ 1&0&-1&0&1&0&1&0\\ -1&0&-1&0&-1&0&1&0\\ -1&0&-1&0&-1&0&1&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0 \end{pmatrix} + \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&-1&0&0&0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0&0&0&0&-1&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{pmatrix}$$

	HL_{80}	HL_{81}	HL_{82}	HL_{83}
HL_{80}	HL_{80}	HL_{81}	HL_{82}	HL_{83}
HL_{81}	HL_{81}	$-HL_{80}$	HL_{83}	$-HL_{82}$
HL_{82}	HL_{82}	$-HL_{83}$	$-HL_{80}$	HL_{81}
HL_{83}	HL_{83}	HL_{82}	$-HL_{81}$	$-HL_{80}$

Figure 8: upper rows: the decomposition of the matrix HL_8 (from Figure 7) as sum of 4 matrices: $HL_8 = HL_{80} + HL_{81} + HL_{82} + HL_{83}$. Bottom row: the multiplication table of these 4 matrices HL_{80}, HL_{81}, HL_{82} and HL_{83}, which is identical to the multiplication table of quaternions by Hamilton. The matrix HL_{80} represents the real unit for this matrix set

The similar situation holds true for the matrix HR_8 (from Figure 7). Figure 9 shows a decomposition of the matrix HR_8 as a sum of 4 matrices: $HR_8 = HR_{80} + HR_{81} + HR_{82} + HR_{83}$. The set of matrices HR_{80}, HR_{81}, HR_{82} and HR_{83} is closed in relation to multiplication and it defines the multiplication table which is identical to the same multiplication table of quaternions by Hamilton. General expression for quaternions in this case can be written as $Q_R = a_0{*}HR_{80} + a_1{*}HR_{81} + a_2{*}HR_{82} + a_3{*}HR_{83}$, where a_0, a_1, a_2, a_3 are real numbers. From this point of view, the (8*8)-genomatrix HR_8 is the quaternion by Hamilton with unit coordinates.

$$HR_8 = HR_{80} + HR_{81} + HR_{82} + HR_{83} =$$

$$
\begin{bmatrix}
0&-1&0&-1&0&1&0&-1\\
0&1&0&1&0&-1&0&1\\
0&1&0&-1&0&-1&0&-1\\
0&-1&0&1&0&1&0&1\\
0&-1&0&1&0&-1&0&-1\\
0&1&0&-1&0&1&0&1\\
0&1&0&1&0&1&0&-1\\
0&-1&0&-1&0&-1&0&1
\end{bmatrix}
=
\begin{bmatrix}
0&-1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&0&-1&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&1
\end{bmatrix}
+
\begin{bmatrix}
0&0&0&-1&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&-1&0&0
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
0&0&0&0&0&1&0&0\\
0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&1\\
0&-1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&-1&0&0&0&0
\end{bmatrix}
+
\begin{bmatrix}
0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&-1&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&-1&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0
\end{bmatrix}
$$

	HR_{80}	HR_{81}	HR_{82}	HR_{83}
HR_{80}	HR_{80}	HR_{81}	HR_{82}	HR_{83}
HR_{81}	HR_{81}	$-HR_{80}$	HR_{83}	$-HR_{82}$
HR_{82}	HR_{82}	$-HR_{83}$	$-HR_{80}$	HR_{81}
HR_{83}	HR_{83}	HR_{82}	$-HR_{81}$	$-HR_{80}$

Figure 9: upper rows: the decomposition of the matrix HR_8 (from Figure 7) as sum of 4 matrices: $H_{8R} = H0_{8R} + H1_{8R} + H2_{8R} + H3_{8R}$. Bottom row: the multiplication table of these 4 matrices HR_{80}, HR_{81}, HR_{82} and HR_{83}, which is identical to the multiplication table of quaternions by Hamilton. HR_{80} represents the real unit for this matrix set

The initial (8*8)-matrix H_8 (Figure 1) can be also decomposed in another way on the base of dyadic-shift decomposition. Figure 10 shows such dyadic-shift decomposition $H_8 = H_{80}+H_{81}+H_{82}+H_{83}+H_{84}+H_{85}+H_{86}+H_{87}$, when 8 sparse matrices H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87} arise (H_{80} is identity matrix). The set H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87} is closed in relation to multiplication and it defines the multiplication table on Figure 10. This multiplication table is identical to the multiplication table of 8-dimensional hypercomplex numbers that are termed as biquaternions by Hamilton (or Hamiltons' quaternions over the field of complex numbers). General expression for biquaternions in this case can be written as $Q_8 = a_0*H_{80}+a_1*H_{81}+a_2*H_{82}+a_3*H_{83}+ a_4*H_{84} +a_5*H_{85}+a_6*H_{86}+a_7*H_{87}$, where a_0, a_1, a_2, $a_{3,}$ a_4, a_5, a_6, a_7 are real numbers. From this point of view, the (8*8)-genomatrix H_8 is Hamiltons' biquaternion with unit coordinates.

$$H_8 = H_{80}+H_{81}+H_{82}+H_{83}+H_{84}+H_{85}+H_{86}+H_{87} =$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{bmatrix} + \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0 \end{bmatrix} +$$

$$\begin{bmatrix} 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{bmatrix}$$

	1	H_{81}	H_{82}	H_{83}	H_{84}	H_{85}	H_{86}	H_{87}
1	**1**	H_{81}	H_{82}	H_{83}	H_{84}	H_{85}	H_{86}	H_{87}
H_{81}	H_{81}	**-1**	H_{83}	$-H_{82}$	H_{85}	$-H_{84}$	H_{87}	$-H_{86}$
H_{82}	H_{82}	H_{83}	**-1**	$-H_{81}$	$-H_{86}$	$-H_{87}$	H_{84}	H_{85}
H_{83}	H_{83}	$-H_{82}$	$-H_{81}$	**1**	$-H_{87}$	H_{86}	H_{85}	$-H_{84}$
H_{84}	H_{84}	H_{85}	H_{86}	H_{87}	**-1**	$-H_{81}$	$-H_{82}$	$-H_{83}$
H_{85}	H_{85}	$-H_{84}$	H_{87}	$-H_{86}$	$-H_{81}$	**1**	$-H_{83}$	H_{82}
H_{86}	H_{86}	H_{87}	$-H_{84}$	$-H_{85}$	H_{82}	H_{83}	**-1**	$-H_{81}$
H_{87}	H_{87}	$-H_{86}$	$-H_{85}$	H_{84}	H_{83}	$-H_{82}$	$-H_{81}$	**1**

Figure 10: Upper rows: the decomposition of the matrix H_8 (from Figure 1) as sum of 8 matrices: $H_8 = H_{80}+H_{81}+H_{82}+H_{83}+H_{84}+H_{85}+H_{86}+H_{87}$. Bottom row: the multiplication table of these 8 matrices H_{80}, H_{81}, H_{82}, H_{83}, H_{84}, H_{85}, H_{86}, H_{87}, which is identical to the multiplication table of biquaternions by Hamilton (or Hamiltons' quaternions over the field of complex numbers). H_{80} is identity matrix

Here for the (8*8)-genomatrix H_8 we have received the interesting result: the sum of two different 4-dimensional quaternions by Hamilton with unit coordinates (they belong to two different matrix representations of Hamiltons' quaternions) generates the 8-dimensional biquaternion with unit coordinates. This result resembles the results, regarding genetic matrices with internal complementarities described above; it resembles a situation when a union of Yin and Yang (or a union of male and female beginnings, or a fusion of male and female gametes) generates a new organism.

3. THE RADEMACHER MATRICES R_4 AND R_8

Now let us pay attention to Rademacher matrices R_4 and R_8 (Figure1) that belong to the second important type of genetic matrices with internal complementarities. Let us initially analyze the matrix R_4, which is the sum of two matrices RL_4 and RR_4 (Figure 11).

$$R_4 = RL_4 + RR_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$RL_4 = RL_{40} + RL_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$RR_4 = RR_{40} + RR_{41} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Figure 11: upper row: the representation of the matrix R_4 as sum of matrices RL_4 and RR_4. Other rows: representations of matrices RL_4 and RR_4 as sums of matrices RL_{40}, RL_{41}, RR_{40} and RR_{41}.

The (4*4)-matrix RL_4 is the sum of two matrices RL_{40} and RL_{41} (Figure 11), the set of which is closed in relation to multiplication and defines the multiplication table of these matrices (Figure 12). This table is identical to the well-known multiplication table of split-complex numbers (their synonyms are Lorentz numbers, hyperbolic numbers, perplex numbers, double numbers, etc. - http://en.wikipedia.org/wiki/Split-complex_number). Split-complex numbers are a two-dimensional commutative algebra over the real numbers.

	RL_{40}	RL_{41}
RL_{40}	RL_{40}	RL_{41}
RL_{41}	RL_{41}	RL_{40}

$$; \; D_L = A_0 * RL_{40} + A_2 * RL_{41} = \begin{bmatrix} A_0 & 0 & A_2 & 0 \\ -A_0 & 0 & -A_2 & 0 \\ A_2 & 0 & A_0 & 0 \\ -A_2 & 0 & -A_0 & 0 \end{bmatrix}$$

$$D_L^{-1} = (A_0^2 - A_2^2)^{-1} * \begin{bmatrix} A_0 & 0 & -A_2 & 0 \\ -A_0 & 0 & A_2 & 0 \\ -A_2 & 0 & A_0 & 0 \\ A_2 & 0 & -A_0 & 0 \end{bmatrix}$$

Figure 12: the multiplication table of two (4*4)-matrices RL_{40} and RL_{41} (Figure 11), which is a set of basic elements of split-complex numbers $D_L = A_0*RL_{40}+A_2*RL_{41}$, where A_0, A_2 are real numbers. The matrix RL_{40} represents the real unit for this matrix set. If $A_0 \neq A2$, the matrix D_L^{-1} is the inverse matrix for D_L by definition on the base of the equation $D_L*D_L^{-1}= RL_{40}$

The set of (4*4)-matrices $D_L = A_0*RL_{40}+A_2*RL_{41}$, where A_0, A_2 are real numbers, represents split-complex numbers in the special (4*4)-matrix form (Figure 12). The classical identity matrix E=[1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1] is absent in the set of matrices D_L, each of which has zero determinant. Consequently the usual notion of the inverse matrix D_L^{-1} (as $D_L*D_L^{-1}$=E) can't be defined in relation to the classical identity matrix E in accordance with the famous theorem about inverse matrices for matrices with zero determinant (Bellman, 1960, Chapter 6, § 4). But the set of matrices D_L has the matrix RL_{40} which possesses all properties of identity matrix (or the real unit) for any member of this set. In the frame of the set of matrices D_L, where the matrix RL_{40} represents the real unity, one can define the special notion of inverse matrix D_L^{-1} for any non-zero matrix D_L in relation to the matrix RL_{40} on the base of equations: $D_L*D_L^{-1} = D_L^{-1}*D_L = RL_{40}$ (Figure 12). From this point of view, the genetic (4*4)-matrix RL_4 is the split-complex number with unit coordinates (A_0=A_2=1). So, we reveal that 4-dimensional spaces can contain 2-parametric subspaces, in which split-complex numbers exist in the form of (4*4)-matrices D_L. It is well known that in mathematics split-complex numbers are traditionally represented in the form of (2*2)-matrix [a_0 a_1; a_1 a_0], where a_0, a_1 are real numbers (http://en.wikipedia.org/wiki/Split-complex_number).

A similar situation holds true for (4*4)-matrices $RR_4 = RR_{40} + RR_{41}$ (from Figure 11). The set of two matrices RR_{40} and RR_{41} is also closed in relation to multiplication; it gives the multiplication table (Figure 13) which is also identical to the multiplication table of split-complex numbers. The set of (4*4)-matrices $D_R = a_1*RR_{40}+a_3*RR_{41}$, where a_1, a_3 are real numbers, represents split-complex numbers in the (4*4)-matrix form (Figure 13). The matrix RR_{40} plays a role of the real unit in this set of matrices D_R. In the case $a_1 \neq a_3$, the matrix D_R^{-1} (Figure 13) is the inverse matrix for D_R by definition on the base of equations $D_R*D_R^{-1} = D_R^{-1}*D_R = RR_{40}$.

	RR_{40}	RR_{41}
RR_{40}	RR_{40}	RR_{41}
RR_{41}	RR_{41}	RR_{40}

; $D_R = A_1*RR_{40}+A_3*RR_{41} =$

0	A_1	0	$-A_3$
0	A_1	0	$-A_3$
0	$-A_3$	0	A_1
0	$-A_3$	0	A_1

$D_R^{-1} = (A_1^2-A_3^2)^{-1}$ *

0	A_1	0	A_3
0	A_1	0	A_3
0	A_3	0	A_1
0	A_3	0	A_1

Figure 13: The multiplication table of two (4*4)-matrices RR_{40} and RR_{41}, which is a set of basic elements of split-complex numbers $D_R = A_1*RR_{40}+A_3*RR_{41}$, where A_1, A_3 are real numbers. The matrix RR_{40} represents the real unit in this matrix set. If $A_1 \neq A_3$, the matrix D_R^{-1} is the inverse matrix for D_R by definition on the base of the equation $D_L*D_L^{-1} = RR_{40}$

The initial matrix R_4 can be also decomposed in another way by means of the dyadic-shift decomposition as it was done for the matrix H_4 on Figure 2. Figure 14 shows such dyadic-shift decomposition $R_4 = R0_4+R1_4+R2_4+R3_4$ when 4 sparse matrices $R0_4$, $R1_4$, $R2_4$ and $R3_4$ arise ($R0_4$ is identity matrix). The set of these matrices $R0_4$, $R1_4$, $R2_4$ and $R3_4$ is closed in relation to multiplication and it defines the multiplication table on Figure 14. This multiplication table is identical to the multiplication table of 4-dimensional hypercomplex numbers that are termed as split-quaternions by J.Cockle and are well known in mathematics and physics (http://en.wikipedia.org/wiki/Split-quaternion). From this point of view, the matrix R_4 is split-quaternion with unit coordinates.

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

	$R0_4$	$R1_4$	$R2_4$	$R3_4$
$R0_4$	$R0_4$	$R1_4$	$R2_4$	$R3_4$
$R1_4$	$R1_4$	$-R0_4$	$R3_4$	$-R2_4$
$R2_4$	$R2_4$	$-R3_4$	$R0_4$	$-R1_4$
$R3_4$	$R3_4$	$R2_4$	$R1_4$	$R0_4$

Figure 14: upper row: the dyadic-shift decomposition $R_4 = R0_4+R1_4+R2_4+R3_4$. Bottom row: the multiplication table of the sparse matrices $R0_4$, $R1_4$, $R2_4$ and $R3_4$, which is identical to the multiplication table of split-quaternions by J.Cockle (http://en.wikipedia.org/wiki/Split-quaternion). $R0_4$ is identity matrix, which plays a role of the real unit in this form of split-quaternions by Cockle.

So we have received the interesting result: the sum of two 2-dimensional split-complex numbers R_{4L} and R_{4R} with unit coordinates (they belong to two different matrix types of split-complex numbers) generates the 4-dimensional split-quaternion with unit coordinates. It resembles again a situation when a union of Yin and Yang (a union of female and male beginnings, or a fusion of male and female gametes) generates a new organism. In particular, it means that the matrix R_4 is one of matrices with internal complementarities.

Let us return now to the (8*8)-matrix R_8 (Figure 1) and demonstrate that it is also a matrix with internal complementarities. Figure 15 shows the matrix R_8 as sum of matrices R_{8L} and R_{8R}.

$R_8 = RL_8 + RR_8 =$

1	0	1	0	1	0	-1	0		0	1	0	1	0	1	0	-1
1	0	1	0	1	0	-1	0		0	1	0	1	0	1	0	-1
-1	0	1	0	-1	0	-1	0		0	-1	0	1	0	-1	0	-1
-1	0	1	0	-1	0	-1	0	+	0	-1	0	1	0	-1	0	-1
1	0	-1	0	1	0	1	0		0	1	0	-1	0	1	0	1
1	0	-1	0	1	0	1	0		0	1	0	-1	0	1	0	1
-1	0	-1	0	-1	0	1	0		0	-1	0	-1	0	-1	0	1
-1	0	-1	0	-1	0	1	0		0	-1	0	-1	0	-1	0	1

Figure 15: the matrix R_8 consists of two complementary parts RL_8 and RR_8

Figure 16 shows a decomposition of the matrix RL_8 (from Figure 15) as a sum of 4 matrices: $RL_8 = RL_{80} + RL_{81} + RL_{82} + RL_{83}$. The set of matrices RL_{80}, RL_{81}, RL_{82} and RL_{83} is closed in relation to multiplication and defines the multiplication table identical to the same multiplication table of split-quaternions by Cockle. General expression for split-quaternions in this case can be written as $S_L = a_0*RL_{80} + a_1*RL_{81} + a_2*RL_{82} + a_3*RL_{83}$, where a_0, a_1, a_2, a_3 are real numbers. From this point of view, the (8*8)-genomatrix RL_8 is split-quaternion by Cockle with unit coordinates.

$$
\begin{bmatrix}
1&0&1&0&1&0&-1&0\\
1&0&1&0&1&0&-1&0\\
-1&0&1&0&-1&0&-1&0\\
-1&0&1&0&-1&0&-1&0\\
1&0&-1&0&1&0&1&0\\
1&0&-1&0&1&0&1&0\\
-1&0&-1&0&-1&0&1&0\\
-1&0&-1&0&-1&0&1&0
\end{bmatrix}
=
\begin{bmatrix}
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0
\end{bmatrix}
+
\begin{bmatrix}
0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
-1&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&-1&0&0&0
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&-1&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0\\
0&0&-1&0&0&0&0&0
\end{bmatrix}
+
\begin{bmatrix}
0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&-1&0\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&-1&0&0&0\\
0&0&-1&0&0&0&0&0\\
0&0&-1&0&0&0&0&0\\
-1&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0
\end{bmatrix}
$$

	RL_{80}	RL_{81}	RL_{82}	RL_{83}
RL_{80}	RL_{80}	RL_{81}	RL_{83}	RL_{83}
RL_{81}	RL_{81}	$-RL_{80}$	RL_{83}	$-RL_{82}$
RL_{82}	RL_{82}	$-RL_{83}$	RL_{80}	$-RL_{81}$
RL_{83}	RL_{83}	RL_{82}	RL_{81}	RL_{80}

Figure 16: Upper rows: the decomposition of the matrix RL_8 (from Figure 15) as sum of 4 matrices: $RL_8 = RL_{80} + RL_{81} + RL_{82} + RL_{83}$. Bottom row: the multiplication table of these 4 matrices RL_{80}, RL_{81}, RL_{82} and RL_{83}, which is identical to the multiplication table of split-quaternions by J.Cockle. RL_{80} represents the real unit for this matrix set

The similar situation holds for the matrix RR_8 (from Figure 15). Figure 17 shows a decomposition of the matrix RR_8 as a sum of 4 matrices: $RR_8 = RR_{80} + RR_{81} + RR_{82} + RR_{83}$. The set of matrices RR_{80}, RR_{81}, RR_{82} and RR_{83} is closed in relation to multiplication and defines the multiplication table that is identical to the same multiplication table of split-quaternions by Cockle. General expression for split-quaternions in this case can be written as $S_R = a_0*RR_{80} + a_1*RR_{81} + a_2*RR_{82} + a_3*RR_{83}$, where a_0, a_1, a_2, a_3 are real numbers. From this point of view, the (8*8)-matrix RR_8 is the split-quaternion with unit coordinates.

$$\begin{array}{|cccccccc|}0&1&0&1&0&1&0&-1\\0&1&0&1&0&1&0&-1\\0&-1&0&1&0&-1&0&-1\\0&-1&0&1&0&-1&0&-1\\0&1&0&-1&0&1&0&1\\0&1&0&-1&0&1&0&1\\0&-1&0&-1&0&-1&0&1\\0&-1&0&-1&0&-1&0&1\end{array} = \begin{array}{|cccccccc|}0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{array} + \begin{array}{|cccccccc|}0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0&0\end{array}$$

$$+ \begin{array}{|cccccccc|}0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&-1&0&0&0&0\end{array} + \begin{array}{|cccccccc|}0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&-1&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&-1&0&0&0&0\\0&0&0&-1&0&0&0&0\\0&-1&0&0&0&0&0&0\\0&-1&0&0&0&0&0&0\end{array}$$

	RR_{80}	RR_{81}	RR_{82}	RR_{83}
RR_{80}	RR_{80}	RR_{81}	RR_{82}	RR_{83}
RR_{81}	RR_{81}	$-RR_{80}$	RR_{83}	$-RR_{82}$
RR_{82}	RR_{82}	$-RR_{83}$	RR_{80}	$-RR_{81}$
RR_{83}	RR_{83}	RR_{82}	RR_{81}	RR_{80}

Figure 17: upper rows: the decomposition of the matrix RR_8 (from Figure 15) as the sum of 4 matrices: $RR_8 = RR_{80} + RR_{81} + RR_{82} + RR_{83}$. Bottom row: the multiplication table of these 4 matrices RR_{80}, RR_{81}, RR_{82} and RR_{83}, which is identical to the multiplication table of split-quaternions by Cockle. RR_{80} represents the real unit here.

The initial (8*8)-matrix R_8 (Figure 1) can be also decomposed in another way by means of the dyadic-shift decomposition as it was done for the matrix H_8 on Figure 10. Figure 18 shows the case of such dyadic-shift decomposition $R_8 = R0_8+R1_8+R2_8+R3_8+R4_8+R5_8+R6_8+R7_8$, when 8 sparse matrices $R0_8$, $R1_8$, $R2_8$, $R3_8$, $R4_8$, $R5_8$, $R6_8$, $R7_8$ arise ($R0_8$ is identity matrix). The set $R0_8$, $R1_8$, $R2_8$, $R3_8$, $R4_8$, $R5_8$, $R6_8$, $R7_8$ is closed in relation to multiplication and defines the multiplication table on Figure 18. This multiplication table is identical to the multiplication table of 8-dimensional hypercomplex numbers that are termed as bi-split-quaternions by Cockle (or split-quaternions over the field of complex numbers). General expression for bi-split-quaternions in this case can be written as $S_8 = a_0*R0_8+a_1*R1_8+a_2*R2_8 +a_3*R3_8+a_4*R4_8$

$+a_5*R5_8+a_6*R6_8+a_7*R7_8$, where a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7 are real numbers. From this point of view, the (8*8)-genomatrix R_8 is bi-split-quaternion with unit coordinates.

$$R_8 = R0_8+R1_8+R2_8+R3_8+R4_8+R5_8+R6_8+R7_8 =$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{bmatrix} + \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0 \end{bmatrix} +$$

$$\begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&-1&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{bmatrix}$$

	$R0_8$	$R1_8$	$R2_8$	$R3_8$	$R4_8$	$R5_8$	$R6_8$	$R7_8$
$R0_8$	$R0_8$	$R1_8$	$R2_8$	$R3_8$	$R4_8$	$R5_8$	$R6_8$	$R7_8$
$R1_8$	$R1_8$	$R0_8$	$R3_8$	$R2_8$	$R5_8$	$R4_8$	$R7_8$	$R6_8$
$R2_8$	$R2_8$	$R3_8$	$-R0_8$	$-R1_8$	$R6_8$	$R7_8$	$-R4_8$	$-R5_8$
$R3_8$	$R3_8$	$R2_8$	$-R1_8$	$-R0_8$	$R7_8$	$R6_8$	$-R5_8$	$-R4_8$
$R4_8$	$R4_8$	$R5_8$	$-R6_8$	$-R7_8$	$R0_8$	$R1_8$	$-R2_8$	$-R3_8$
$R5_8$	$R5_8$	$R4_8$	$-R7_8$	$-R6_8$	$R1_8$	$R0_8$	$-R3_8$	$-R2_8$
$R6_8$	$R6_8$	$R7_8$	$R4_8$	$R5_8$	$R2_8$	$R3_8$	$R0_8$	$R1_8$
$R7_8$	$R7_8$	$R6_8$	$R5_8$	$R4_8$	$R3_8$	$R2_8$	$R1_8$	$R0_8$

Figure 18: Upper rows: the decomposition of the matrix R_8 (from Figure 1) as sum of 8 matrices: $R_8 = R0_8+R1_8+R2_8+R3_8+R4_8+R5_8+R6_8+R7_8$. Bottom row: the multiplication table of these 8 matrices $R0_8$, $R1_8$, $R2_8$, $R3_8$, $R4_8$, $R5_8$, $R6_8$ and $R7_8$, which is identical to the multiplication table of bi-split-quaternions by Cockle. $R0_8$ is identity matrix and represents the real unit here.

Here for the (8*8)-genomatrix R_8 we have received the interesting result: the sum of two different 4-dimensional split-quaternions by Cockle with unit coordinates (they belong to two different matrix types of split-quaternion numbers) generates the 8-dimensional bi-split-quaternion with unit coordinates. This result resembles the above-described result about the sum of 2-dimensional split-complex numbers with unit coordinates that generates the 4-dimensional split-quaternion with unit coordinates (Figures 12-14). It also resembles a situation when a union of Yin and Yang (a union of male and female beginnings or a fusion of male and female gametes) generates a new organism.

4. MATRICES OF GENETIC DUPLETS AND TRIPLETS

Theory of noise-immunity coding is based on matrix methods. For example, matrix methods allow transferring high-quality photos of Mar's surface via millions of kilometers of strong interference. In particularly, Kronecker families of Hadamard matrices are used for this aim. Kronecker multiplication of matrices is the well-known operation in fields of signals processing technology, theoretical physics, etc. It is used for transition from spaces with a smaller dimension to associated spaces of higher dimension.

By analogy with theory of noise-immunity coding, the 4-letter alphabet of RNA (adenine A, cytosine C, guanine G and uracil U) can be represented in a form of the (2*2)-matrix [C U; A G] (Figure 19) as a kernel of the Kronecker family of matrices $[C\ U;\ A\ G]^{(n)}$, where (n) means a Kronecker power (Figure 19). Inside this family, this 4-letter alphabet of monoplets is connected with the alphabet of 16 duplets and 64 triplets by means of the second and third Kronecker powers of the kernel matrix: $[C\ U;\ A\ G]^{(2)}$ and $[C\ U;\ A\ G]^{(3)}$, where all duplets and triplets are disposed in a strict order (Figure 19). We begin with the alphabet A, C, G, U of RNA here because of mRNA-sequences of triplets define protein sequences of amino acids in a course of its reading in ribosomes (below we will separately consider the case of DNA with its own alphabet).

Figure 19 contains not only 64 triplets but also amino acids and stop-codons encoded by the triplets in the case of the Vertebrate mitochondrial genetic code that is the most symmetrical among known variants of the genetic code (http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi). One can see on Figure 19 that in the matrix $[C\ U;\ A\ G]^{(3)}$ the set of columns with even numeration 0, 2, 4, 6 and the set of columns with odd numeration 1, 3, 5, 7 have the same collection of amino acids and stop-codons. In other words, the nature has constructed the distribution of amino acids and stop-codons in accordance with the principle of the matrix with internal complementarity. This fact is only one of evidences that the described matrices with internal complementarities are the mathematical patterns of the genetic coding system (the mathematical partners of the genetic code).

Let us explain black-and-white mosaics of [C U; A G]$^{(2)}$ and [C U; A G]$^{(3)}$ (Figure 19) which reflect important features of the genetic code. These features are connected with a specificity of reading of mRNA-sequences in ribosomes to define protein sequences of amino acids (this is the reason, why we use the alphabet A, C, G, U of RNA in matrices on Figure 19; below we will consider the case of DNA-sequences separately).

C	U
A	G

CC	CU	UC	UU
CA	CG	UA	UG
AC	AU	GC	GU
AA	AG	GA	GG

CCC **Pro**	CCU **Pro**	CUC **Leu**	CUU **Leu**	UCC **Ser**	UCU **Ser**	UUC **Phe**	UUU **Phe**
CCA **Pro**	CCG **Pro**	CUA **Leu**	CUG **Leu**	UCA **Ser**	UCG **Ser**	UUA **Leu**	UUG **Leu**
CAC **His**	CAU **His**	CGC **Arg**	CGU **Arg**	UAC **Tyr**	UAU **Tyr**	UGC **Cys**	UGU **Cys**
CAA **Gln**	CAG **Gln**	CGA **Arg**	CGG **Arg**	UAA **Stop**	UAG **Stop**	UGA **Trp**	UGG **Trp**
ACC **Thr**	ACU **Thr**	AUC **Ile**	AUU **Ile**	GCC **Ala**	GCU **Ala**	GUC **Val**	GUU **Val**
ACA **Thr**	ACG **Thr**	AUA **Met**	AUG **Met**	GCA **Ala**	GCG **Ala**	GUA **Val**	GUG **Val**
AAC **Asn**	AAU **Asn**	AGC **Ser**	AGU **Ser**	GAC **Asp**	GAU **Asp**	GGC **Gly**	GGU **Gly**
AAA **Lys**	AAG **Lys**	AGA **Stop**	AGG **Stop**	GAA **Glu**	GAG **Glu**	GGA **Gly**	GGG **Gly**

Figure 19: the first three representatives of the Kronecker family of RNA-alphabetic matrices [C U; A G]$^{(n)}$. Black color marks 8 strong duplets in the matrix [C U; A G]$^{(2)}$ (at the top) and 32 triplets with strong roots in the matrix [C U; A G]$^{(3)}$ (bottom). 20 amino acids and stop-codons, which correspond to triplets, are also shown in the matrix [C U; A G]$^{(3)}$ for the case of the Vertebrate mitochondrial genetic code

A combination of letters on the two first positions of each triplet is ususally termed as a "root" of this triplet (Konopelchenko, Rumer, 1975a,b; Rumer, 1968). Modern science recognizes many variants (or dialects) of the genetic code, data about which are shown on the NCBI's website http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi. 17 variants (or dialects) of the genetic code exist that differ one from another by some details of correspondences between triplets and objects encoded by them. Most of these dialects (including the so called Standard Code and the Vertebrate Mitochondrial Code) have the symmetrologic general scheme of these correspondences, where 32 "black" triplets with "strong roots" and 32 "white" triplets with "weak" roots exist (see details in

(Petoukhov, 2008c). In this basic scheme, the set of 64 triplets contains 16 subfamilies of triplets, every one of which contains 4 triplets with the same two letters on the first positions (an example of such subsets is the case of four triplets CAC, CAA, CAT, CAG with the same two letters CA on their first positions). In the described basic scheme, the set of these 16 subfamilies of *NN*-triplets is divided into two equal subsets. The first subset contains 8 subfamilies of so called "two-position" *NN*-triplets, a coding value of which is independent on a letter on their third position: (CCC, CCT, CCA, CCG), (CTC, CTT, CTA, CTG), (CGC, CGT, CGA, CGG), (TCC, TCT, TCA, TCG), (ACC, ACT, ACA, ACG), (GCC, GCT, GCA, GCG), (GTC, GTT, GTA, GTG), (GGC, GGT, GGA, GGG). An example of such subfamilies is the four triplets CGC, CGA, CGT, CGC, all of which encode the same amino acid Arg, though they have different letters on their third position. The 32 triplets of the first subset are termed as "triplets with strong roots" (Konopelchenko, Rumer, 1975a,b; Rumer, 1968). The following duplets are appropriate 8 strong roots for them: CC, CT, CG, AC, TC, GC, GT, GG (strong duplets). All members of these 32 *NN*-triplets and 8 strong duplets are marked by black color in the matrices $[C\ U;\ A\ G]^{(3)}$ and $[C\ U;\ A\ G]^{(2)}$ on Figures 19.

The second subset contains 8 subfamilies of "three-position" NN-triplets, the coding value of which depends on a letter on their third position: (CAC, CAT, CAA, CAG), (TTC, TTT, TTA, TTG), (TAC, TAT, TAA, TAG), (TGC, TGT, TGA, TGG), (AAC, AAT, AAA, AAG), (ATC, ATT, ATA, ATG), (AGC, AGT, AGA, AGG), (GAA, GAT, GAA, GAG). An example of such subfamilies is the four triplets CAC, CAA, CAT, CAC, two of which (CAC, CAT) encode the amino acid His and the other two (CAA, CAG) encode another amino acid Gln. The 32 triplets of the second subset are termed as "triplets with weak roots" (Konopelchenko, Rumer, 1975a,b; Rumer, 1968). The following duplets are appropriate 8 weak roots for them: CA, AA, AT, AG, TA, TT, TG, GA (weak duplets). All members of these 32 *NN*-triplets and 8 weak duplets are marked by white color in the matrices $[C\ U;\ A\ G]^{(3)}$ and $[C\ U;\ A\ G]^{(2)}$ on Figure 19.

From the point of view of its black-and-white mosaic, each of columns of genetic matrices $[C\ U;\ A\ G]^{(2)}$ and $[C\ U;\ A\ G]^{(3)}$ has a meander-like character and coincides with one of Rademacher functions that form orthogonal systems and well known in discrete signals processing. These functions contain elements "+1" and "-1" only. Due ti this fact, one can construct Rademacher representations of the symbolic genomatrices $[C\ U;\ A\ G]^{(2)}$ and $[C\ U;\ A\ G]^{(3)}$ (Figure 19) by means of the following operation: each of black duplets and of black triplets is replaced by number "+1" and each of white duplets and white triplets is replaced by number "-1". This operation leads immediately to the matrices R_4 and R_8 from Figure 1, that are the Rademacher representations of the phenomenological genomatrices [C U; A G]$^{(2)}$ and [C U; A G]$^{(3)}$. This fact is one of evidences of algebraic nature of the genetic code.

One can note that genomatrices [C U; A G]$^{(2)}$ and [C U; A G]$^{(3)}$ and their Rademacher representations R_4 and R_8 (Figure 1) are connected on the base of the equations (1), where ⊗ means Kronecker multiplication:

$$R_4 \otimes [1\ 1;\ 1\ 1] = R_8, \quad [C\ U;\ A\ G]^{(2)} \otimes [C\ U;\ A\ G] = [C\ U;\ A\ G]^{(3)} \qquad (1)$$

Here [1 1; 1 1] is the traditional (2*2)-matrix representation of split-complex number with unit coordinates, that can be considered as the Rademacher representation R_2 of the genomatrix [C U; A G]. The equations (1) testify that, in the case of RNA-alphabet, each of its four letters in the matrix [C U; A G] should be taken as equal to number "+1": A=C=G=U=+1. They also show that Rademacher representations R_2 and R_4 of matrices [C U; A G] and [C U; A G]$^{(2)}$ can be considered as basic due to the fact that the Rademacher representation R_8 is deduced from them by means of their Kronecker multiplication.

Now let us pay attention to the DNA alphabet (adenine A, cytosine C, guanine G and thymine T) and the appropriate Kronecker family of matrices [C T; A G]$^{(n)}$. What kind of black-and-white mosaics (or a disposition of elements "+1" and "-1" in numeric representations of these symbolic matrices) can be appropriate in this case for the basic matrix [C T; A G] and [C T; A G]$^{(2)}$? The important phenomenological fact is that the thymine T is a single nitrogenous base in DNA which is replaced in RNA by another nitrogenous base U (uracil) for unknown reason (this is one of the mysteries of the genetic system). In other words, in this system the letter T is the opposition in relation to the letter U, and so the letter T can be symbolized by number "-1" (instead of number "+1" for U). By this objective reason, one can construct numeric representations H_2 and H_4 of mentioned matrices [C T; A G] and [C T; A G]$^{(2)}$ by means of the following algorithm of transformation of black-and-white mosaics of matrices [C U; A G] and [C U; A G]$^{(2)}$ from Figure 19 together with their Rademacher representations R_2 and R_4:
- in matrices [C T; A G] and [C T; A G]$^{(2)}$, each of monoplets and duplets that begin with the letter T, should be taken with opposite color in comparison with appropriate entries in matrices [C U; A G] and [C U; A G]$^{(2)}$ from Figure 19; correspondingly numeric representations of these DNA-alphabetic matrices [C T; A G] and [C T; A G]$^{(2)}$ reflect the new mosaics of these symbolic matrices.

The numeric representation H_8 of the DNA-alphabetic matrix of triplets $[C\ T;\ A\ G]^{(3)}$ is constructed on the base of equations (2) by analogy with equations (1):

$$H_4 \otimes [1\ -1;\ 1\ 1] = H_8, \quad [C\ T;\ A\ G]^{(2)} \otimes [C\ T;\ A\ G] = [C\ T;\ A\ G]^{(3)} \qquad (2)$$

Here [1 -1; 1 1] is the traditional (2*2)-matrix representation of complex number with unit coordinates. The black-and-white mosaic of the matrix $[C\ T;\ A\ G]^{(3)}$ is defined by the disposition of numbers "+1" and "-1" in its numeric representation H_8. Figure 20 shows DNA-alphabetic matrices [C T; A G], $[C\ T;\ A\ G]^{(2)}$ and $[C\ T;\ A\ G]^{(3)}$ with their mosaics constructed by this way, which is based on the objective properties of the molecular-genetic system and can be used in biological computers of organisms. One can see that mosaics of these symbolic matrices $[C\ T;\ A\ G]^{(2)}$ and $[C\ T;\ A\ G]^{(3)}$ coincide with the disposition of numbers "+1" and "-1" in numeric matrices H_4 and H_8 (Figure 1) that can be termed as "Hadamard representations" of these genomatrices because matrices H_4 and H_8 satisfy the definition of Hadamard matrices (Petoukhov, 2008b, 2011).

C	T
A	G

CC	CT	TC	TT
CA	CG	TA	TG
AC	AT	GC	GT
AA	AG	GA	GG

;

CCC	CCT	CTC	CTT	TCC	TCT	TTC	TTT
CCA	CCG	CTA	CTG	TCA	TCG	TTA	TTG
CAC	CAT	CGC	CGT	TAC	TAT	TGC	TGT
CAA	CAG	CGA	CGG	TAA	TAG	TGA	TGG
ACC	ACT	ATC	ATT	GCC	GCT	GTC	GTT
ACA	ACG	ATA	ATG	GCA	GCG	GTA	GTG
AAC	AAT	AGC	AGT	GAC	GAT	GGC	GGT
AAA	AAG	AGA	AGG	GAA	GAG	GGA	GGG

Figure 20: the first three representatives [C T; A G], $[C\ T;\ A\ G]^{(2)}$ and $[C\ T;\ A\ G]^{(3)}$ of the Kronecker family of DNA-alphabetic matrices $[C\ T;\ A\ G]^{(n)}$. Hadamard representations H_4 and H_8 of the symbolic matrices $[C\ T;\ A\ G]^{(2)}$ and $[C\ T;\ A\ G]^{(3)}$ with the same mosaics are shown on Figure 1

Genetic matrices with internal complementarities resemble objects with Yin and Yang parts from doctrines of Ancient China. One can add here the following mathematical fact. The famous Yin-Yang symbol ☯ has a symmetrical configuration: its 180-degree turn changes only its black-and-white mosaic, but the new configuration of the symbol coincides with the initial. It is interesting that the 180-degree turn of the genetic matrices R_4, R_8, H_4, H_8 (Figure 1) leads to a similar result: mosaics of these matrices are essentially changed but the new matrices are again matrices with internal complementarities, algebraic properties of which coincide with the initial (the same multiplication tables as on Figures 9, 10, 12-14, 16-18). So, the mythological object allows revealing new mathematical properties of the genetic matrices in this case.

Phenomenology of the genetic system gives additional confirmations of its connection with the mosaic genomatrices [C T; A G]$^{(n)}$, numeric representations of which posess internal complementarities. In matrices [C T; A G]$^{(n)}$, let us enumerate their 2^n columns from left to right by numbers 0, 1, 2, .., 2^n-1 and then consider two sets of n-plets (oligonucleotides) in each of matrices [C T; A G]$^{(n)}$: 1) the first set contains all n-plets from columns with even numeration 0, 2, 4, … (this set is conditionally termed as the even-set or the Yin-set); 2) the second set contains all n-plets from columns with odd numeration 1, 3, 5, … (this set is conditionally termed as the odd-set or the Yang-set).

For example, the genomatrix [C T; A G]$^{(3)}$ (Figure 19) contains the even-set of 32 triplets in its columns with even numerations 0, 2, 4, 6 (CCC, CCA, CAC, CAA, ACC, ACA, AAC, AAA, CTC, CTA, CGC, CGA, ATC, ATA, AGC, AGA, TCC, TCA, TAC, TAA, GCC, GCA, GAC, GAA, TTC, TTA, TGC, TGA, GTC, GTA, GGC, GGA) and the odd-set of 32 triplets in its columns with odd numerations 1, 3, 5, 7 (CCT, CCG, CAT, CAG, ACT, ACG, AAT, AAG, CTT, CTG, CGT, CGG, ATT, ATG, AGT, AGG, TCT, TCG, TAT, TAG, GCT, GCG, GAT, GAG, TTT, TTG, TGT, TGG, GTT, GTG, GGT, GGG). One can show, for example, that the structure of the whole human genome is connected with the equal devision of the whole set of 64 triplets into the even-set of 32 triplets and the odd-set of 32 triplets. Really, let us calculate total quantities (frequencies F_{even} and F_{odd}) of members of these two sets of triplets in the whole human genome that contains the huge number 2.843.411.612 (about three billion) triplets. The initial data about this genome (Figure 21) are taken by the author from the article (Perez, 2010). Very different frequencies of different triplets are represented in this genome. For example, the frequency of the triplet CGA is equal to 6.251.611 and the frequency of the triplet TTT is equal to 109.591.342; they differ in 18 times approximately. But our result of the calculation shows that the total quantities of members of the even-set (F_{even}) and of the odd-set (F_{odd}) in the whole human genome are equal to each other with a precision within 0,12%:

F_{even} = 1.420.853.821 for the even-set of 32 triplets;

F_{odd} = 1.422.557.791 for the odd-set of 32 triplets.

One should note that the work (Perez, 2010, Table 10) shows another variant of division of the set of 64 triplets into two other subsets with 32 triplets in each not on the basis of the matrix approach but on the base of using a traditional table of triplets and a principle of “codons and their mirror-codons”. This variant also reveals an approximate equality of quantities of members of these two subsets with a high precision for the case of the whole human genome.

TRIPLET	TRIPLET FREQUENCY	TRIPLET	TRIPLET FREQUENCY	TRIPLET	TRIPLET FREQUENCY	TRIPLET	TRIPLET FREQUENCY
AAA	109143641	CAA	53776608	GAA	56018645	TAA	59167883
AAC	41380831	CAC	42634617	GAC	26820898	TAC	32272009
AAG	56701727	CAG	57544367	GAG	47821818	TAG	36718434
AAT	70880610	CAT	52236743	GAT	37990593	TAT	58718182
ACA	57234565	CCA	52352507	GCA	40907730	TCA	55697529
ACC	33024323	CCC	37290873	GCC	33788267	TCC	43850042
ACG	7117535	CCG	7815619	GCG	6744112	TCG	6265386
ACT	45731927	CCT	50494519	GCT	39746348	TCT	62964984
AGA	62837294	CGA	6251611	GGA	43853584	TGA	55709222
AGC	39724813	CGC	6737724	GGC	33774033	TGC	40949883
AGG	50430220	CGG	7815677	GGG	37333942	TGG	52453369
AGT	45794017	CGT	7137644	GGT	33071650	TGT	57468177
ATA	58649060	CTA	36671812	GTA	32292235	TTA	59263408
ATC	37952376	CTC	47838959	GTC	26866216	TTC	56120623
ATG	52222957	CTG	57598215	GTG	42755364	TTG	54004116
ATT	71001746	CTT	56828780	GTT	41557671	TTT	109591342

Figure 25: quantities of repetitions of each triplet in the whole human genome (from [Perez, 2010])

More general confirmation of genetic importance of the structure of genomatrices with internal complementarities for long nucleotide sequences was revealed by the results of the study of the Symmetry Principle № 6 from the work (Petoukhov, 2008c, 6th version, section 11), where a special notion of fractal genetic nets for long nucleotide sequences were used in contrast to this article. Now we propose to use the relevant phenomenologic data for justification and development of the new idea: the described matrices with internal complementarities are important algebraic patterns for structurization of the genetic coding system, the nature of which has algebraic bases.

The described connection between the genetic system and matrices with internal complementarities is associated with the Plato's conception about androgynes. In accordance with this ancient conception, in primal times people had doubled bodies. But at one moment the gods have punished them by splitting them in half. Ever since that time, people run around saying they are looking for their other half because they are really trying to recover their primal nature (http://en.wikipedia.org/wiki/Symposium_(Plato). This conception is frequently used in discussions on important facts of embryology and other modern scientific fields about hermaphroditism including the embryological principle of primordial hermaphroditism, etc. (Dreger, 1998; Money, 1990, etc.). Taking the Plato's conception into account, genetic matrices with internal complementarities can be also termed as "androgynous matrices". Results of our researches lead to the idea that phenomena of hermaphroditism have a basic analogue at the molecular-genetic level. These results can be related with biological problems of genetically inherited symmetries and dissymmetry (Darvas, 2007; Gal, 2011; Hellige, 1993).

5. SOME CONCLUDING REMARKS

In the beginning of 19-th century, there was a belief was about the existence of one arithmetic that is true for all natural systems. But after the discovery of quaternions by Hamilton, the science has been compelled to refuse the former belief about existence of only one true arithmetic/algebra in the world (see (Kline, 1980)). It has recognized, that various natural systems can have not only their own geometry (Euclidean or non-Euclidean geometries), but also their own algebra (arithmetic of multi-dimensional numbers). If the scientist takes inadequate algebra to model a natural system, he/she can repeat the impressive example by Hamilton, who has wasted 10 years to solve the task of 3D space transformations on the bases of inadequate 3-dimensional algebras (this task needs the 4-dimensional algebra of Hamilton's quaternions). Modern theoretical physics includes, as one of its main parts, a great number of attempts to reveal what kinds of multi-dimensional numeric systems correspond to ensembles of relations in concrete physical systems.

The results of our researches discover that relations in the genetic coding system correspond to the described algebraic system of matrices with internal complementarities. If the researcher does not take into account this fact and this special mathematics, he/she runs the risk of wasting a lot of time and effort because of the application of inadequate approaches to study algebraic properties of the genetic system.

In particularly, this article shows the connection of the genetic coding system with quaternions by Hamilton. Hamilton quaternions are closely related to the Pauli matrices, the theory of the electromagnetic field (Maxwell wrote his equation on the language of Hamilton quaternions), the special theory of relativity, the theory of spins, quantum theory of chemical valency, etc. In the twentieth century thousands of works were devotes to quaternions in physics [http://arxiv.org/abs/math-ph/0511092]. Now Hamilton quaternions are manifested in the genetic code system. Our scientific direction

- "matrix genetics" - has led to the discovery of an important bridge among physics, biology and computer science for their mutual enrichment. In addition, our study provides a new example of the inconceivable effectiveness of mathematics: abstract mathematical structures derived by mathematicians at the tip of the pen 160 years ago, are embodied inside the molecular-genetic system which is the informational basis of living matter. And the fact that mathematics is opened by means of painful reflection (like Hamilton, who has spent 10 years of continuous thought to discover his quaternions) is already represented in the genetic coding system.

The described genetic matrices with internal complementarities (or "androgenous matrices") posess many other interesting mathematical properties related to cyclic and dyadic shifts, multiplications of these matrices, Kronecker families of matrices $R_4 \otimes [1\ 1;\ 1\ 1]^{(n)}$ and $H_4 \otimes [1\ {-1};\ 1\ 1]^{(n)}$, dichotomous trees of different 2^n-dimensional numbers, rotational transformations of these numeric genomatrices into new numeric genomatrices with internal complementarities, etc. A set of (2^n*2^n)-matrices with internal complementarities contains a huge quantity of different types of matrix representations of complex numbers (or relevant algebraic fields (http://en.wikipedia.org/wiki/Field_(mathematics)) and of split-complex numbers that didn't specially studied in mathematics previously, as the author can judge. The relevant 2^n-dimensional numeric systems, including the said plurality of complex and split-complex numbers and their extensions, have perspectives to be applied in mathematical natural sciences and signals processing. Here one can remember the statement: "*Profound study of nature is the most fertile source of mathematical discoveries*" (Fourier, 2006). The discovery of genetic importance of matrices with internal complementarities gives us a possibility to divide sets of amino acids and stop-signals in interesting sub-sets in accordance with the structure of the genomatrix $[C\ T;\ A\ G]^{(3)}$; it also presents new approaches to study proteins. One should note that phenomena of complementarities play a basic role at different genetic levels. We are hoping to expend this and similar topics in future publications.

The notion of number is one of the main notions of mathematics. In a long evolution of this notion, many kinds of multi-dimensional numerical systems have appeared. Complex numbers and split-complex numbers occupy a particularly important place in mathematics and mathematical natural sciences. For example, complex numbers have appeared as magic instruments for development of theories and calculations in the field of problems of heat, light, sounds, vibrations, elasticity, gravitation, magnetism, electricity, liquid streams, and phenomena of a micro-world. These complex numbers are mathematical basis of quantum mechanics and of many other branches of sciences. For example, the Schrödinger equation contains the imaginary unit, and the wave functions of quantum mechanics are complex-valued. This article shows that many kinds of complex numbers and split-complex numbers exist, which are connected with the genetic matrices. One can think that this splitting of numeric basis of mathematical natural sciences lead to a relevant splitting in mathematical natural sciences. For example, one can ask what kinds of complex numbers should be used in the Schrödinger equation? Or can different types of wave functions of quantum mechanics exist, which correspond to different kinds of complex numbers? In our opinion, such questions should be deeply analyzed in future.

This article proposes a new mathematical approach to study "a partnership between genes and mathematics" (see Section 1 above). In the author's opinion, this kind of mathematics is beautiful and it can be used for further developing of algebraic biology and theoretical physics in accordance with the famous statement by P.Dirac, who taught that a creation of a physical theory must begin with the beautiful mathematical theory: "*If this theory is really beautiful, then it necessarily will appear as a fine model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics*" (Arnold, 2007). According to Dirac, all new physics, including relativistic and quantum, are developing in this way.

Results of matrix genetics lead to the idea that the structure of the genetic coding system is dictated by patterns of described numeric genomatrices; here one can remember the famous Pythagorean statement that "numbers rule the world" with the refinement that we should talk now about multi-dimensional numbers.

Acknowledgments. Described researches were made by the author in the frame of a long-term cooperation between Russian and Hungarian Academies of Sciences. The author is grateful to Darvas G., Stepanyan I.V., Svirin V.I. for their collaboration.

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