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SYMMETRIES, GENERALIZED NUMBERS AND HARMONIC LAWS IN MATRIX GENETICS

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Abstract. *This paper is devoted to the presentation and analysis of matrix representations of the genetic code. Principal attention is paid to a family of the genetic matrices which are constructed on the basis of Gray code ordering of their rows and columns. This Gray code ordering reveals new connections of the genetic code to: 8-dimensional bipolar algebras; Hadamard matrices; golden matrices; Pythagorean musical scale, and an integer triangle attributed to Nicomachus, a Syrian mathematician from second century A.D. All of these mathematical entities possess symmetrical features. Some questions about silver means and Pythagorean triples are also described by these genetic matrices and their generalizations.*

Keywords: genetic code, Gray code, multidimensional algebra, Hadamard matrices, golden section, Pythagorean musical scale, bisymmetric matrices, silver means, Pythagorean triples.

1. INTRODUCTION.

The achievements of bioinformatics have led to new thoughts on the essence of life. "*Life is a partnership between genes and mathematics*" (Stewart, 1999). But what kind of mathematics relates to the genetic code and what kind of mathematics lies behind genetic phenomenology which accounts for the great noise-immunity of the genetic code? This question is one of the main challenges in the mathematical natural sciences today.

The genetic code serves to provide information transfer with a high-level of noise-immunity along a chain of generations. In view of this we will look for genetic mathematics in the formalisms of the theory of discrete signal processing, noise-immunity codes, and computer informatics. This report carries forth the authors' investigations [Kappraff, 2000a-c; Petoukhov, 2001, 2005, 2008].

Genetic coding possesses noise-immunity. Mathematical theories of noise-immunity coding and discrete signal processing are based on matrix methods for the representation and analysis of information. These matrix methods, endowed with symmetry, are applied to the analysis of ensembles of molecular elements of the genetic code. This report describes a matrix representation of the genetic code connected with Gray code which is widely used in the field of information coding. Gray code is a system of enumerating integers using 0's and 1's such that from one integer to the next, a single bit changes from 1 to 0 or 0 to 1 and that change occurs in the least significant digit to result in a value not previously obtained. This Gray code method of presentation, which differs significantly from the previous representation based on Kronecker families of matrices [Petoukhov, 2001, 2005], reveals new connections of the genetic code to the following mathematical entities: 8-dimensional bipolar algebras; Hadamard matrices; golden matrices; Pythagorean musical scale, and the Nicomachus triangle. This mathematics is rich in symmetry. These matrices will also be shown to describe all of the silver means and Pythagorean triples based on work by Kappraff and Adamson [Kappraff, 2009].

2. A SPECIAL MATRIX PRESENTATION OF THE GENETIC CODE.

We present the genetic alphabet encoding the DNA/RNA bases, A (adenine), C (cytosine), G (guanine) and U in the form of an alphabetic matrix S:

	1	0
1	C	A
0	U	G

The first column of this matrix contains pyrimidines, C and U, each of which is marked by the symbol 1 taking into account this mutual chemical aspect. The second column contains purines, A and G, each of which is marked by the symbol 0 taking into account this mutual chemical aspect. The first row of the matrix contains C and A, each of which is keto from the viewpoint of their physiochemical significance and is marked by the symbol 1. The second row of the matrix contains U and G, each of which is amino from the viewpoint of their physiochemical significance and is marked by the symbol 0. In this way each of these nitrogenous bases obtains a binary expression in accordance with its position inside the matrix S: C is expressed by 11; A is expressed by 10; U is expresses by 01; G is expressed by 00.

This paper considers these binary symbols from the perspective of Gray code. In Gray code the numbers 0, 1, 2, 3, 4, 5, 6, 7 are represented as follows: 0 - 000; 1 – 001; 2 – 011; 3 – 010; 4 – 110; 5 – 111; 6 – 101; 7 – 100. One can construct a (4*4)-matrix S_2 for the 16 genetic duplets and a (8*8)-matrix S_3 for the 64 genetic triplets, where all rows and columns are ennumerated in ascending order in Gray code (Figure 1). Each element of these matrices is coded by the bits in its row and column, e.g., the duplet UC is represented by 0111 because U is symbolized by 01 and C is symbolized by 11).

	10 (3)	**11 (2)**	**01 (1)**	**00 (0)**
10 (3)	**CG**	**CU**	**AU**	**AG**
11 (2)	**CA**	**CC**	**AC**	**AA**
01 (1)	**UA**	**UC**	**GC**	**GA**
00 (0)	**UG**	**UU**	**GU**	**GG**

	100(7)	**101(6)**	**111(5)**	**110(4)**	**010(3)**	**011(2)**	**001(1)**	**000(0)**
100 (7)	**CGG (Arg)**	**CGU (Arg)**	**CUU (Leu)**	**CUG (Leu)**	**AUG (Met)**	**AUU (Ile)**	**AGU (Ser)**	**AGG (Stop)**
101 (6)	**CGA (Arg)**	**CGC (Arg)**	**CUC (Leu)**	**CUA (Leu)**	**AUA (Met)**	**AUC (Ile)**	**AGC (Ser)**	**AGA (Stop)**
111 (5)	**CAA (Gln)**	**CAC (His)**	**CCC (Pro)**	**CCA (Pro)**	**ACA (Thr)**	**ACC (Thr)**	**AAC (Asn)**	**AAA (Lys)**
110 (4)	**CAG (Gln)**	**CAU (His)**	**CCU (Pro)**	**CCG (Pro)**	**ACG (Thr)**	**ACU (Thr)**	**AAU (Asn)**	**AAG (Lys)**
010 (3)	**UAG (Stop)**	**UAU (Tyr)**	**UCU (Ser)**	**UCG (Ser)**	**GCG (Ala)**	**GCU (Ala)**	**GAU (Asp)**	**GAG (Glu)**
011 (2)	**UAA (Stop)**	**UAC (Tyr)**	**UCC (Ser)**	**UCA (Ser)**	**GCA (Ala)**	**GCC (Ala)**	**GAC (Asp)**	**GAA (Glu)**
001 (1)	**UGA (Trp)**	**UGC (Cys)**	**UUC (Phe)**	**UUA (Leu)**	**GUA (Val)**	**GUC (Val)**	**GGC (Gly)**	**GGA (Gly)**
000 (0)	**UGG (Trp)**	**UGU (Cys)**	**UUU (Phe)**	**UUG (Leu)**	**GUG (Val)**	**GUU (Val)**	**GGU (Gly)**	**GGG (Gly)**

Figure 1. The matrix S_2 for 16 duplets (on the left side) and the matrix S_3 for 64 triplets with the Gray code ordering of numeration of all rows and columns. A correspondence between triplets and amino acids is shown for the case of the vertebrate mitochondrial genetic code. Names of amino acids are given by their standard abbreviations. The black-and-white mosaic is explained in the text.

Matrix S_3 in Figure 1 shows, additionally, what kind of amino acids (or a stop-signal) is encoded by each triplet of the vertebrate mitochondrial genetic code, which is the most symmetrical known dialect of genetic coding. It should be explained that modern science knows many dialects of the genetic code, as shown on the NCBI's website http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi. By tradition, natural objects with the greatest symmetry are studied first after which cases in which symmetry is violated are studied. Similarly, the authors first investigate the vertebrate mitochondrial genetic code which is the most symmetrical. Its should be noted that some authors consider this dialect to be not only the most "perfect" and symmetrical but also the most ancient although this opinion is subject to some debate.

Let us explain a black-and-white mosaic of the matrix S_3 in Figure 1. Figure 2 shows the correspondence between the set of 64 triplets, sometimes referred to as codons, and the set of 20 amino acids with stop-signals (Stop) of protein synthesis in the vertebrate mitochondrial genetic code.

8 subfamilies of the "two-position" NN-triplets and the amino acids, which are encoded by them	**8 subfamilies of the "three-position" NN-triplets and the amino acids, which are encoded by them**
CCC, CCA, CCU, CCG ➔ Pro	CAC, CAA, CAU, CAG ➔ His, Gln
CUC, CUA, CUU, CCG ➔ Leu	AAC, AAA, AAU, AAG ➔ Asn, Lys
CGC, CGA, CGU, CGG ➔ Arg	AUC, AUA, AUU, AUG ➔ Ile, Met
ACC, ACA, ACU, ACG ➔ Thr	AGC, AGA, AGU, AGG ➔ Ser, Stop
UCC, UCA, UCU, UCG ➔ Ser	UAC, UAA, UAU, UAG ➔ Tyr, Stop
GCC, GCA, GCU, GCG ➔ Ala	UUC, UUA, UUU, UUG ➔ Phe, Leu
GUC, GUA, GUU, GUG ➔ Val	UGC, UGA, UGU, UGG ➔ Cys, Trp
GGC, GGA, GGU, GGG ➔ Gly	GAC, GAA, GAU, GAG ➔ Asp, Glu

Figure 2. The case of the vertebrate mitochondrial genetic code. The initial data were taken from the NCBI's web-site http://www.ncbi.nlm.nih.gov/Taxonomy/Utils/wprintgc.cgi.

The set of 64 triplets contains such 16 subfamilies of triplets, every one of which contains 4 triplets with the same pair of letters in the first two positions of each triplet (an example of such subsets is the case of the four triplets CAC, CAA, CAU, CAG with the same two letters CA on their first two positions). We shall refer to such subfamilies as NN-triplets. In the case of the vertebrate mitochondrial code, the set of these 16 subfamilies of NN-triplets is divided into two equal subsets from the point of view of degeneration properties of the code (Figure 2). The first subset contains 8 subfamilies of so called "two-position" NN-triplets, a coding value of which is independent of a letter on their third position. An example of such subfamilies is the four triplets CGC, CGA, CGU, CGC, all of which encode the same amino acid Arg, although they have different letters on their third position. All members of such subfamilies of NN-triplets are marked by black color in the genomatrix on Figure 1.

The second subset contains 8 subfamilies of "three-position" NN-triplets, a coding value which depends on a letter on their third position. An example of such subfamilies is the four triplets CAC, CAA, CAU, CAC, two of which (CAC, CAU) encode the amino acid His and other two (CAA, CAG) encode another amino acid Gln. All members of such subfamilies of NN-triplets are marked by white color in the genomatrix on Figure 1. So this genomatrix has 32 black triplets and 32 white triplets. Each subfamily of four NN-triplets appears in an appropriate (2x2)-subquadrant of the genomatrix. The black triplets encode the so-called high-degeneracy amino acids composed of four or more triplets, and from Figure 2 it can be seen that there are eight such amino acids, whereas the white triplets encode low-degeneracy amino acids with only two triplets, and it can be seen from Figure 2 that there are twelve such amino acids [Petoukhov, 2005]. The black-and-white mosaic of the matrix S_3 (Figure 1) has inversion symmetry relative to the matrix center. This mosaic reflects the specificity of degeneracy of the vertebrate mitochondrial genetic code.

3. CONCERNING AN ALGORITHMIC CONNECTION OF THE GENETIC (8*8)-MATRIX WITH AN 8-DIMENSIONAL ALGEBRA.

Is the mosaic genetic matrix S_3 (Figure 1) connected with a matrix form of representation of a multi-dimensional algebra? Yes, a positive answer was received on the basis of a simple algorithm of digitization which uses the molecular characteristics of the nitrogenous bases A, C, G, U/T from the genetic alphabet. The genomatrix S_3 (Figure 1) is transformed into a new numeric 8-parametric matrix Y_8 (Figure 3) by means of this alphabetic algorithm.

$Y_8 =$

	100(7)	101(6)	111(5)	110(4)	010(3)	011(2)	001(1)	000(0)
100 (7)	x_1	x_0	x_2	x_3	$-x_7$	$-x_6$	$-x_4$	$-x_5$
101 (6)	x_1	x_0	x_2	x_3	$-x_7$	$-x_6$	$-x_4$	$-x_5$
111 (5)	$-x_3$	$-x_2$	x_0	x_1	x_5	x_4	$-x_6$	$-x_7$
110 (4)	$-x_3$	$-x_2$	x_0	x_1	x_5	x_4	$-x_6$	$-x_7$
010 (3)	$-x_7$	$-x_6$	x_4	x_5	x_1	x_0	$-x_2$	$-x_3$
011 (2)	$-x_7$	$-x_6$	x_4	x_5	x_1	x_0	$-x_2$	$-x_3$
001 (1)	$-x_5$	$-x_4$	$-x_6$	$-x_7$	x_3	x_2	x_0	x_1
000 (0)	$-x_5$	$-x_4$	$-x_6$	$-x_7$	x_3	x_2	x_0	x_1

Figure 3. *The matrix Y_8, the black cells of which contain coordinates with the sign "+" and the white cells of which contain coordinates with the sign "-". The numeration of the columns and the rows is identical to the numeration of the columns and the rows of the genomatrix S_3 of Figure 1.*

The cells of the matrix Y_8, which are occupied by elements with the sign "+", are marked by dark color. The cells of the matrix Y_8, which are occupied by components with the sign "-", are marked by white color. Such a black-and-white mosaic of the matrix Y_8 is identical to the black-and-white mosaic of the genomatrix on Figure 1. The matrix Y_8 has the 8 independent parameters x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, which are interpreted as real numbers here. It has been discovered that the matrix Y_8 is the matrix form of a special 8-dimensional algebra over the field of real numbers.

Let us describe the alphabetic algorithm for the digitization of 64 triplets [Petoukhov, 2008a,b]. This algorithm is based on utilizing the two following binary-oppositional attributes of the genetic letters A, C, G, U/T: "purine or pyrimidine" and "2 or 3" hydrogen bonds. It uses also the well-known thesis of molecular genetics that different positions within triplets have different code meanings. For example the article [Konopelchenko, & Rumer, 1975] has described that the first two positions of each triplet form "the root of the codon" and that they differ drastically in their function from the third position and by its special role. In view of this "alphabetic" algorithm, the transformation of genomatrix S_3 into matrix Y_8 is not an abstract and arbitrary action at all, but such a transformation can actually be utilized by a kind of bio-computer system within organisms.

1) The alphabetic algorithm of the digitization defines the special scheme of reading each triplet: the first two positions of the triplet are read by genetic systems from the perspective of one molecular attribute and the third position of the triplet is read from the perspective of another molecular

attribute. By this alphabetic algorithm, which allows one to recode the symbolic matrix S_3 into the numeric matrix Y_8 (Figure 3), each triplet is read in the following way: Two first positions of each triplet are marked by the symbol "α" in place of the complementary letters C and G on these positions and by the symbol "β" in place of the complementary letters A and U;

2) The third position of each triplet is marked by the symbol "γ" in place of the pyrimidines (C or U) at this position and by the symbol "δ" in place of the purines (A or G);
3) The triplets, which have the letters C or G in their first position, receive the sign "-" in only those instances for which their second position is occupied by the letter A. The triplets, which have the letters A or U in their first position, receive the sign "+" in those cases only for which their second positions are occupied by the letter C.

For example, the triplet CAG receives the symbol "-αβδ", because its first letter C is symbolized by "α", its second letter A is symbolized by "β", and its third letter G is symbolized by "δ". This triplet possesses the sign "-" because its first position has the letter C and its second position has the letter A. One can see that this algorithm recodes all triplets from the traditional alphabet C, A, U, G into the new alphabet α, β, γ, δ. As a result, each triplet receives one of the following 8 expressions: $\alpha\alpha\gamma = x_0$, $\alpha\alpha\delta = x_1$, $\alpha\beta\gamma = x_2$, $\alpha\beta\delta = x_3$, $\beta\alpha\gamma = x_4$, $\beta\alpha\delta = x_5$, $\beta\beta\gamma = x_6$, $\beta\beta\delta = x_7$. We will suppose that the symbols "α", "β", "γ", "δ" are real numbers. This algorithm transforms the initial symbolic matrix S_3 (Figure 1) into the numeric matrix Y_8 with the 8 coordinates x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7 (Figure 3) for x_i real, which we shall refer as the "Y-coordinates".

One can represent the 8-dimensional matrix Y_8 (Figure 3) as the sum of the 8 basis matrices $\mathbf{j}_k$ (k = 0, 1, 2,…7), each of which is connected with one of the coordinates. In this case one can present the matrix Y_8 by the vector form of Equation 1, rewritten explicitly in Figure 4:

$$Y_8=x_0*\mathbf{j}_0+x_1*\mathbf{j}_1+x_2*\mathbf{j}_2+x_3*\mathbf{j}_3+x_4*\mathbf{j}_4+x_5*\mathbf{j}_5+x_6*\mathbf{j}_6+x*\mathbf{j}_7 \qquad (1)$$

$$\boldsymbol{Y_8 = x_0*}\begin{bmatrix} 0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0 \end{bmatrix} + \boldsymbol{x_1*}\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{bmatrix} +$$

$$+\boldsymbol{x_2*}\begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix} + \boldsymbol{x_3*}\begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0 \end{bmatrix} +$$

$$+\boldsymbol{x_4*}\begin{bmatrix} 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0 \end{bmatrix} + \boldsymbol{x_5*}\begin{bmatrix} 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0 \end{bmatrix} +$$

$$+x_6*\begin{bmatrix} 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1&0\\ 0&-1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0 \end{bmatrix} + x_7*\begin{bmatrix} 0&0&0&0&-1&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&-1\\ -1&0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{bmatrix}$$

Figure 4. The representation of matrix Y_8 as the sum of the 8 basis matrices. The left column shows the basis matrices, which are related to the coordinates x_0, x_2, x_4, x_6 with the even indexes. The right column shows the basic matrices, which are related to the coordinates x_1, x_3, x_5, x_7 with the odd indexes.

The most important and unexpected feature of these matrices is that the set of these 8 basis matrices $\mathbf{j_0}$, $\mathbf{j_1}$, $\mathbf{j_2}$, $\mathbf{j_3}$, $\mathbf{j_4}$, $\mathbf{j_5}$, $\mathbf{j_6}$, $\mathbf{j_7}$ forms a closed set under multiplication: a multiplication between any pair of matrices from this set generates another matrix from this set. A multiplication table for these products is shown in Figure 5. The result of multiplying any two basis elements, which are taken from the left column and the upper row, is shown in the cell at the intersection of its row and column in the multiplication table (for example, in accordance with this multiplication table $\mathbf{j_2}*\mathbf{j_5} = -\mathbf{j_7}$). It should be noted that this multiplication is not commutative.

	j_0	j_1	j_2	j_3	j_4	j_5	j_6	j_7
j_0	j_0	j_1	j_2	j_3	j_4	j_5	j_6	j_7
j_1	j_0	j_1	j_2	j_3	j_4	j_5	j_6	j_7
j_2	j_2	j_3	$-j_0$	$-j_1$	$-j_6$	$-j_7$	j_4	j_5
j_3	j_2	j_3	$-j_0$	$-j_1$	$-j_6$	$-j_7$	j_4	j_5
j_4	j_4	j_5	j_6	j_7	j_0	j_1	j_2	j_3
j_5	j_4	j_5	j_6	j_7	j_0	j_1	j_2	j_3
j_6	j_6	j_7	$-j_4$	$-j_5$	$-j_2$	$-j_3$	j_0	j_1
j_7	j_6	j_7	$-j_4$	$-j_5$	$-j_2$	$-j_3$	j_0	j_1

Figure 5. The multiplication table of the basic matrices $\mathbf{j_0}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}, \mathbf{j_4}, \mathbf{j_5}, \mathbf{j_6}, \mathbf{j_7}$ of the matrix Y_8 from Figures 3 and 4.

Such a multiplication table defines an 8-dimensional algebra Y_8 over a field. Multiplication of any two members of the octet algebra Y_8 generates a new member of the same algebra. Concerning the matrix form of such multiplication, it means that both factors in a multiplication have the identical matrix disposition of their 8 parameters x_0, x_1, …, x_7 (in the first factor) and y_0, y_1,…, y_7 (in the second factor) and the final matrix has the same matrix disposition of its 8 relevant parameters z_0, z_1, …, z_7. This is similar to the situation of real numbers (or of complex numbers, or of hypercomplex numbers) when multiplication of any two members of the numeric system generates a new member of the same numerical system. In other words, the expression $Y_8=x_0*\mathbf{j_0}+x_1*\mathbf{j_1}+x_2*\mathbf{j_2}+x_3*\mathbf{j_3}+x_4*\mathbf{j_4}+x_5*\mathbf{j_5}+x_6*\mathbf{j_6}+x_7*\mathbf{j_7}$ can be considered to be a kind of 8-dimensional number similar to the way in which complex numbers are 2-dimensional or quarterions are 4-dimensional.

This new genetic algebra Y_8 belongs to a set of bipolar algebras (or Yin-Yang-algebras, or even-odd-algebras) and it has interesting mathematical properties which are similar in many aspects to genetic bipolar algebras described in the works [Petoukhov, 2008a,b].

Permutations of elements play an important role in the theory of signal processing. One can study the influence that the simultaneous permutations of the positions in all 64 genetic triplets have upon matrices S_3 and Y_8. Six permutations of triplets are possible: 1-2-3, 2-3-1, 3-1-2, 1-3-2, 2-1-3, 3-2-1. For example, if one changes the initial order, 1-2-3, in all triplets into the new order, 2-3-1, then many cells of the initial genomatrix S_3 are occupied by new triplets. For example, the matrix cell with the triplet CAU is occupied by the triplet AUC, etc. As a result, the initial genomatrix $S_{3/123}$ (we have included here an additional index 123 into the previous symbol for matrix S_3 to show the appropriate juxtaposition of the 1-2-3 positions in the triplets) is reconstructed into the new genomatrix $S_{3/231}$ with a new black-and-white mosaic. This new matrix $S_{3/231}$ is transformed by the same alphabetic algorithm of the digitization which was described above into a new integer matrix $Y_{8/231}$ with the same eight coordinates x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7 in the new configuration. One can check that this new numeric matrix $Y_{8/231}$ is the matrix form of the representation of a new 8-dimensional bipolar algebra. The same holds for the other four matrices $S_{3/312}$, $S_{3/132}$, $S_{3/213}$ and $S_{3/321}$, which also correspond to new configurations of triplets juxtaposed by the permutations: 312, 132, 213, 321. These four matrices, each of which has its own black-and-white mosaic, are transformed by the same alphabetic digitization algorithm into new 8-fold matrices $Y_{8/312}$, $Y_{8/132}$, $Y_{8/213}$ and $Y_{8/321}$. Each of these four matrices also provides a matrix form of representation of its own 8-dimensional bipolar algebra. This matrix genetics approach has many connections with principles and methods of symmetry (see for example [Darvas, 2007; Darvas, Petoukhov, 2005]).

4. A CONNECTION TO HADAMARD MATRICES.

Let us consider some connections of the genetic matrices to matrix formalisms from the theory of discrete signal processing. One of the most important classes of matrices associated with this theory is the so called Hadamard matrices. These matrices are also used in the study of error-correcting codes such as the Reed-Muller code, in spectral analysis, in multi-channel spectrometers with Hadamard transformations, in quantum computers with Hadamard gates (or logical operators), in quantum mechanics as unitary operators, etc. A huge number of scientific publications are devoted to Hadamard matrices. These matrices provide for effective means of information processing.

By definition, a Hadamard matrix of dimension "n" is the (n*n)-matrix H(n) with elements "+1" and "-1". It satisfies the condition $H(n)*H(n)^T = n*I_n$, where $H(n)^T$ is the transposed matrix and I_n is the (n*n)-identity matrix. The Hadamard matrices of dimension 2^k are given by the recursive formula $H(2^k) = H(2)^{(k)} = H(2)\otimes H(2^{k-1})$ for $2 \le k\in N$, where $\otimes$ denotes the Kronecker (or tensor) product, (k) denotes Kronecker exponentiation, k and N are integers, H(2) is illustrated in Figure 6.

$$\boldsymbol{H(2)} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} ; \quad \boldsymbol{H(4)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\boldsymbol{H(2^K)} = \begin{bmatrix} H(2^{K-1}) & H(2^{K-1}) \\ -H(2^{K-1}) & H(2^{K-1}) \end{bmatrix}$$

Figure 6. The family of Hadamard matrices $H(2^k)$ based on the Kronecker product

Is there any natural connection between the genetic matrices $S_{3/123}$, $S_{3/231}$, $S_{3/312}$, $S_{3/132}$, $S_{3/213}$ and $S_{3/321}$ and Hadamard matrices? A positive answer to this question could lead to new thoughts about the structure of the genetic code.

The answer to this question is positive: such an algorithmic connection does exist. It is associated with fundamental and somewhat enigmatic features of the genetic code, namely, 1) the mutual replacement of the letters U and T in RNA and DNA and, 2) the difference of these letters from other letters A, C, G due to the absence of amids (amino-groups) within them. This algorithm is named the U-algorithm, and it was used in the study of genomatrices of the Kronecker type [Petoukhov, 2008a,b]. Let us demonstrate its application to the genomatrix $S_{3/123}$ (Figure 1).

CGG (Arg)	CGU (Arg)	CUU (Leu)	CUG (Leu)	AUG (Met)	AUU (Ile)	AGU (Ser)	AGG (Stop)
CGA (Arg)	CGC (Arg)	CUC (Leu)	CUA (Leu)	AUA (Met)	AUC (Ile)	AGC (Ser)	AGA (Stop)
CAA (Gln)	CAC (His)	CCC (Pro)	CCA (Pro)	ACA (Thr)	ACC (Thr)	AAC (Asn)	AAA (Lys)
CAG (Gln)	CAU (His)	CCU (Pro)	CCG (Pro)	ACG (Thr)	ACU (Thr)	AAU (Asn)	AAG (Lys)
UAG (Stop)	UAU (Tyr)	UCU (Ser)	UCG (Ser)	GCG (Ala)	GCU (Ala)	**GAU (Asp)**	GAG (Glu)
UAA (Stop)	UAC (Tyr)	UCC (Ser)	UCA (Ser)	GCA (Ala)	GCC (Ala)	GAC (Asp)	GAA (Glu)
UGA (Trp)	UGC (Cys)	UUC (Phe)	UUA (Leu)	GUA (Val)	GUC (Val)	GGC (Gly)	GGA (Gly)
UGG (Trp)	UGU (Cys)	UUU (Phe)	UUG (Leu)	GUG (Val)	GUU (Val)	GGU (Gly)	GGG (Gly)

1	-1	-1	1	-1	1	1	-1
1	1	1	1	-1	-1	-1	-1
-1	-1	1	1	1	1	-1	-1
-1	1	-1	1	1	-1	1	-1
1	-1	1	-1	1	-1	1	-1
1	1	-1	-1	1	1	-1	-1
1	1	1	1	1	1	1	1
1	-1	-1	1	1	-1	-1	1

*Figure 7. Top: the result of the transformation of matrix $S_{3/123}$ into a new mosaic genomatrix. Bottom: a Hadamard (8*8)-matrix with the similar mosaic.*

According to the U-algorithm we invert the signs in cells of the matrix $S_{3/123}$ every time the letter U occupies the first or the third positions of a triplet. For example, by this U-algorithm the cells with the triplets UCA and GAU change their sign once, while the cell with the triplet UAU changes its sign twice which means that the sign of this cell is unchanged. As a result of such U-algorithmic sign changes, a new mosaic genomatrix appears (Figure 7, top). Its mosaic is identical to the corresponding mosaic of a Hadamard matrix. Actually, if each black triplet (white triplet) in this genomatrix is replaced by number "+1" ("-1") a numeric matrix is formed (Figure 7, bottom). One can easily check that this matrix satisfies the definition of Hadamard (8*8)-matrices: $H(8)*H(8)^T = 8*I_n$.

The five genomatrices $S_{3/231}$, $S_{3/312}$, $S_{3/132}$, $S_{3/213}$ and $S_{3/321}$ are also connected with Hadamard matrices because they are transformed into their own Hadamard matrices by means of the same U-algorithm. One can suppose that this U-algorithm (of inverting the signs every time the letter U or T appears in an odd position of triplets) is connected with the biological mechanism of mutual replacement of the letters U and T at transition from RNA to DNA and vice versa.

5. GENOMATRICES WITH NUMBERS OF HYDROGEN BONDS.

Each nitrogenous base has a particular number of its hydrogen bonds: the complementary bases, C and G, have 3 hydrogen bonds, while the complementary bases, A and U, have 2 hydrogen bonds. Investigations of the Kronecker family of genomatrices reveals interesting mathematical properties of these matrices in the case for which a symbol of each triplet is replaced by the number of ways in which the hydrogen bonds of its bases can interact given by the product of its number of hydrogen bonds. For example the triplet CAG is replaced by an integer 3*2*3=18, etc. In what follows, we study the integer based genomatrices formed by these integer product replacements as ordered by Gray code.

As a result, the symbolic genomatrix S_2, S_3, etc. (Figure 1) are transformed into a family of integer matrices M_2, M_3, etc. (Figure 8). These matrices are bisymmetric that is symmetric relative to both diagonals.

$$M_1 = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}; \quad M_2 = \begin{vmatrix} 9 & 6 & 4 & 6 \\ 6 & 9 & 6 & 4 \\ 4 & 6 & 9 & 6 \\ 6 & 4 & 6 & 9 \end{vmatrix}; \quad M_3 = \begin{vmatrix} 27 & 18 & 12 & 18 & 12 & 8 & 12 & 18 \\ 18 & 27 & 18 & 12 & 8 & 12 & 18 & 12 \\ 12 & 18 & 27 & 18 & 12 & 18 & 12 & 8 \\ 18 & 12 & 18 & 27 & 18 & 12 & 8 & 12 \\ 12 & 8 & 12 & 18 & 27 & 18 & 12 & 18 \\ 8 & 12 & 18 & 12 & 18 & 27 & 18 & 12 \\ 12 & 18 & 12 & 8 & 12 & 18 & 27 & 18 \\ 18 & 12 & 8 & 12 & 18 & 12 & 18 & 27 \end{vmatrix}$$

Figure 8. Integer matrices with elements represented by integer products of hydrogen bonds in monoplets, duplets and triplets

Notice that in M_1 the natural numbers 2, 3 appear, in M_2 the numbers 4, 6, 9 appear while in M_3 the numbers 8, 12, 18, 27 appear, with each row and column expressing the same species of positive integers with no integer appearing adjacent to itself in a row or column. These sets of integers come from a triangle of positive integers attributed to the 2nd century AD Syrian mathematician Nicomachus (Kappraff, 2000) and represents sequences of musical fifths. The Nicomachus Triangle, T(n,k), is reproduced in Table 1 where the integers in the n-th row are $\{2^{n-k}3^k, 0 \le k \le n\}; n \ge 0.$ Here if the central integer 6 is thought to be the length of a string representing a fundamental tone, then 4 and 9 of row 3 represent the string lengths corresponding to the rising and falling musical fifths, 2:3 and 3:2. Also the fifth row represents the string lengths that give rise to a pentatonic scale with fundamental string length of 36 units while the integers in row 7 represent string lengths of a heptatonic scale with 216 as the string length of the fundamental. The Triangle T(n,k) in Table 1 has the property that every row, column, diagonal, and line joining any two elements contains a geometric progression.

Table 1. The Nicomachus Triangle, T(n,k)

1						
2	3					
4	6	9				
8	12	18	27			
16	24	36	54	81		
32	48	72	108	162	243	
64	96	144	216	324	486	729
		etc.				

Table 2. Pascal's Triangle

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
	etc.			

T(n,k) is the triangle of coefficients in the expansion of $(2 + 3x)^n$; given by the generating function $\dfrac{1}{1-y(2+3x)}$. For example, 8,12,18,27 are generated by $(2+3x)^3 = 8 + 3\times 12x + 3\times 18x^2 + 27x^3$ where we see that there is one 8, one 27, three 12's and three 18's in each row or column of matrix M_3. Furthermore if we set x = 1 we find that the sum of the elements in each row or column of M_n equals 5^n, e.g., for M_3 the sum = 125. In other words, successive integers from a row of the Nicomachus triangle are multiplied by successive integers from rows of Pascal's Triangle, given in Table 2, e.g., (1, 3, 3, 1) $\bullet$ (8, 12, 18, 27) where $\bullet$ denotes dot product in order to sum the row and column elements of M_3.

It was shown [Petoukhov, 2001, 2005] that,

$$P_1 = M_1^{1/2} = \begin{bmatrix} \tau & 1/\tau \\ 1/\tau & \tau \end{bmatrix} \tag{2}$$

where $\tau = \dfrac{1+\sqrt{5}}{2}$ is the golden mean. Associating τ with 11 and 00, and $1/\tau$ with 10 and 01, we obtain matrices of the square roots of each of the higher M_n matrices denoted by P_n. For example,

$$P_2 = M_2^{1/2} = \begin{bmatrix} \tau^2 & 1 & 1/\tau^2 & 1 \\ 1 & \tau^2 & 1 & 1/\tau^2 \\ 1/\tau^2 & 1 & \tau^2 & 1 \\ 1 & 1/\tau^2 & 1 & \tau^2 \end{bmatrix} \tag{3}$$

1010 corresponds to $\tau \times \tau = \tau^2$, 1100 corresponds to $1/\tau \times 1/\tau = 1/\tau^2$, and 1011 corresponds to $\tau \times 1/\tau = 1$. It can be shown that the elements of $P_n = M_n^{1/2}$ are all powers of the golden mean [Kapraff, 2009].

6. GENERALIZED BISYMMETRIC MATRICES.

Bisymmetric matrices are investigated in matrix genetics specially [Petoukhov, 2001, 2005]. Here we study general forms of bisymmetric matrices, which are constructed by means of the Gray code ordering. We start with the matrix M_1:

$$M_1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \tag{4}$$

The higher order matrices M_n are determined in a similar manner as was done for the matrix with a = 3 and b=2. They contain columns and rows with integers from each row of the generalized Nicomachus Triangle in Table 3 with multiplicities given by Pascal's Triangle. For example, using elements of row 3 of Table 3,

$$M_2 = \begin{bmatrix} a^2 & ab & b^2 & ab \\ ab & a^2 & ab & b^2 \\ b^2 & ab & a^2 & ab \\ ab & b^2 & ab & a^2 \end{bmatrix} \tag{5}$$

Table 3. Generalized Nicomachus Triangle

$$\begin{array}{lllll} 1 \\ b & a \\ b^2 & ab & a^2 \\ b^3 & ab^2 & a^2b & a^3 \\ b^4 & ab^3 & a^2b^2 & a^3b & a^4 \end{array} \tag{6}$$

The elements of M_n are generated by $(b + ax)^n$ with each row and column of M_n summing to $(a+b)^n$. On the other hand we have shown [Kappraff, 2009] that the square root of the M_1 and M_2 matrices can be expressed as,

$$P_1 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} \alpha^2 & \alpha\beta & \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 & \alpha\beta & \beta^2 \\ \beta^2 & \alpha\beta & \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 & \alpha\beta & \alpha^2 \end{bmatrix} \tag{7a,b}$$

where,

$$\alpha = \frac{b}{2\beta} \quad \text{and} \quad \beta = \sqrt{\frac{a - \sqrt{a^2 - b^2}}{2}}\ . \tag{8a,b}$$

Therefore α and β are real for a>b and complex for a<b. Since $P_1^2 = M_1$, it follows that,

$$a = \alpha^2 + \beta^2, \quad b = 2\alpha\beta \ , \tag{9a,b}$$

and from this it follows that,

$$\alpha + \beta \ = \sqrt{a+b}\ , \text{and } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{2a}{b} \tag{10a,b}$$

Also, α and β are roots of the fourth degree polynomial,

$$x^4 - (\alpha^2 + \beta^2)x^2 + \alpha^2\beta^2 = 0 \tag{11}$$

Making use of Equations 8a,b, Equation 11 is rewritten,

$$x^4 - ax^2 + \frac{b^2}{4} = 0 \tag{12}$$

where using Equation 8a and 9a,

$$a = \frac{b^2}{4\beta^2} + \beta^2 \tag{13}$$

Equation 12 can also be rewritten as,

$$x^4 = ax^2 - \frac{b^2}{4}\ , \tag{14}$$

and if we consider the geometric sequence,

$$1,\ \alpha^2,\ \alpha^4,\ \alpha^6,\ \alpha^8, \ldots \tag{15}$$

where ,

$$\alpha^{2n} = a\alpha^{2n-2} \ - \frac{b^2}{4}\alpha^{2n-4}\ , \tag{16}$$

this corresponds to a generalized "Fibonacci" sequence, $\{c_n\}$ in which,

$$c_1 = 0, c_2 = 1 \ \text{ and } \ c_n = ac_{n-1} - \frac{b^2}{4}c_{n-2}\ . \tag{17a}$$

The ratio of successive terms, $\frac{c_{n+1}}{c_n}$ approaches α^2 in the limit where, in the case that α is complex, then α^2 denotes the square of the absolute value. Also $\frac{c_{n+1}}{c_n}$ approaches α^2 from below.

If we let $g_n = c_n^2 - c_{k-1}c_{k-2}$, then g_n can be shown to satisfy the recursion,

$$g_n = a^2 g_{n-1} + (\frac{b^4}{16} - \frac{a^2b^2}{4})g_{n-2}. \tag{17b}$$

Setting b = 2 and letting $a = N^2 \pm 2$, Equation 13 reduces to,

$$\frac{1}{\beta^2} + \beta^2 = N^2 \pm 2 \quad \text{and} \quad \alpha = 1/\beta. \qquad (18a,b)$$

.
It follows that,

$$\frac{1}{\beta} \mp \beta = N \qquad (19a)$$

Since $\alpha = 1/\beta$, Equation 19a is rewritten,

$$\alpha \mp \frac{1}{\alpha} = N \qquad (19b)$$

We refer to solutions of the equations,

$$x - 1/x = N \quad \text{and} \quad x + 1/x = N \qquad (20)$$

as the N-th silver means of the first and second kind respectively and denote them as $SM_1(N)$ and $SM_2(N)$ [Kappraff, 2000b]. When N = 1, $x = SM_1(1) = \tau$, the golden mean. Therefore, in Equation 19b, $\alpha = SM_1(N)$ or $\alpha = SM_2(N)$.

As a result of Equation 20, α satisfies one of the equations,

$$x^2 = Nx \pm 1 \qquad (21)$$

Therefore, the sequence

$$1,\ \alpha^1,\ \alpha^2,\ \alpha^3,\ \alpha^4,\ \ldots \qquad (22)$$

is a generalized Pell sequence [Kappraff, 2000b] and satisfies the recursion,

$$\alpha^n = N\alpha^{n-1} \pm \alpha^{n-2} \qquad (23)$$

as does the sequence, $\{c_k\}$ where,

$$c_k = Nc_{k-1} \pm c_{k-2} \qquad (24a)$$

where $\frac{c_{n+1}}{c_n}$ approaches α in the limit. We also find that when b = 2, Equation 17b has the special solution: $g_n = k,$ i.e.,

$$c_n^2 - c_{k-1}c_{k-2} = k \quad \text{for all n} \qquad (24b)$$

This means that if k = 0, the sequence $\{c_n\}$ is a geometric sequence. Otherwise it is an approximate geometric sequence.

We consider eight examples:

Example 1: a = 3, b = 2, N=1. $\alpha = SM_1(1) = \tau$ and $\beta = 1/\tau$, row and column elements are generated by $(2+3x)^n$, row and column sum = 5^n, Sequence

17 yields {0,1,3,8,21,…}, even indexed Fibonacci terms with ratio of successive terms approaching τ^2. In Equation 24b, we find that k = 1. The golden mean has found many applications. LeCorbusier made it the basis of his Modulor system of architectural design (Kappraff, 2000c).

Example 2: a = 6 , b = 2, N = 2. Replacing this into Equation 8 yields $\alpha = SM_1(2) = 1+\sqrt{2}$, $\beta = \frac{1}{1+\sqrt{2}}$, row and column elements are generated by $(2+6x)^n$, row and column sum = 8^n Sequence 17a yields: {0,1,6,35,204,…}, approaching β^2. In Equation 24b, we find that k = 1. The proportion, $1+\sqrt{2}$ is commonly known as the silver mean and was the basis of the system of proportions used in the Roman empire (Kappraff, 2000c).

Example 3: a = 5, b = 4, $\alpha = 2$, $\beta = 1$, row and column elements generated by $(4 + 5x)^n$, row and column sum = 9^n. Sequence 17, i.e., $c_n = 5c_{n-1} - 4c_{n-2}$, yields : {0,1,5,21,85,341,1365,…} = $\{c_n = \frac{4^n-1}{3}\}$ as the ratio of successive terms tends to 4.

Example 4: a = 5, b = 3, $\alpha = 3/\sqrt{2}$, $\beta = 1/\sqrt{2}$, row and column elements generated by $(5+3x)^n$, row and column sum = 8^n. The generalized Nicomachus Triangle in Table 4 is generated from $\{3^{n-k} \times 5^k\}(0 \le k \le n)$.

Table 4. Generalized Nicomachus Triangle Generated by (3,5)

1				
3	5			
9	15	25		
27	45	75	125	
81	135	225	375	625

Each column of this Triangle represents a sequence of musical fifths and recreates the ancient Pythagorean scale, whereas any three successive columns generate the tones of the ancient Just scale [Kappraff, 2000, 2009; McClain, 1976].

Example 5: a = 4, b = 3, $\alpha = \sqrt{\frac{4+\sqrt{7}}{2}}$, $\beta = \sqrt{\frac{4-\sqrt{7}}{2}}$, row and column elements generated by $(4+3x)^n$, row and column sum = 7^n.

Example 6: a = 7, b = 2, N = 3, $\alpha = SM_2(3) = \tau^2$, $\beta = 1/\tau^2$, row and column elements generated by $(2+7x)^n$, row and column sum = 9^n.

Example 7: a = 1, b = 1, $\alpha = 1/\sqrt{2}$, $\beta = 1/\sqrt{2}$. Row and column elements are generated by $(1+x)^n$.

All elements of the generalized Nicomachus Triangle (see Table 3) are ones but taking into account multiplicity yields Pascal's Triangle (see Table 2) whose (n,k)-th element is equal to $\frac{n!}{k!(n-k)!}$.

The ratio of successive terms in Sequence 17, i.e., $c_1 = 0, c_2 = 1$ and $c_n = c_{n-1} - \frac{1}{4}c_{n-2}$, yields: 1, ¾, 2/3, 5/8, 3/5, 7/12, 4/7, … approaching the value of ½. These ratios are the fundamental, musical fourth, fifth, minor sixth, major sixth of the ancient Just scale, and two approximations to the major and minor sevenths, all approaching the octave value of ½. If the modulus M of a pair of successive approximating fractions a/b and c/d is defined as M = (ad – bc) then all moduli of the approximating sequence have the value 1, e.g., (1x4 – 1x3) = 1, (3x3 – 4x2) = 1, etc. As a result, the approximating fractions appear as elements of successive rows of the Farey Table to the right of ½ [Kappraff, 2000b].

Example 8: a = 1, b = 2, N = i, $\alpha = SM_1(i) = e^{i\pi/6}$, $\beta = e^{-i\pi/6}$, row and column terms are generated by $(2+x)^n$, row and column sum = 3^n.
The generalized Nicomachus Triangle yields,

Table 5. Generalized Nicomachus Triangle Generated by (1,2)

1				
2	1			
4	2	1		
8	4	2	1	
16	8	4	2	1

Multiplying the elements of Table 5 by the elements of Pascal's Triangle to account for multiplicity yields the square of Pascal's Triangle, a triangle whose (i,j)-th entry is (i,j)x2^{i-j} where (i,j) is the element of the i-th row and j-th column of Pascal's Triangle.

Table 6. Square of Pascal's Triangle

1				
2	1			
4	4	1		
8	12	6	1	
16	32	24	8	1

The rows of Table 6 give the number of vertices, edges, faces, cells, etc of hypercubes, of increasing dimension, e.g., H_0 (point) V = 1; H_1 (line segment) V = 2, E = 1; H_2 (square) V = 4, E = 4, F=1; H_3 (cube) V = 8, E = 12, F = 6, C = 1; H_4 (tesseract) V = 16, E = 32, F = 24, C = 8, hypercube = 1. Sequence 17 generates the sequence: 0,1,1,0,-1,-1,0,1,1,…, and the ratio of successive terms should approach α^2 = 1. In fact, the ratio of a subsequence approaches 1 identically as it should. In Eq. 24b, we find once again that k = 1.

If a = 2, b = 1, then the generalized Nicomachus Triangle is identical to the one for a = 1, b = 2 but the columns are in reverse order. On the other hand $\alpha = \sqrt{\frac{2+\sqrt{3}}{2}}$ and $\beta = \sqrt{\frac{2-\sqrt{3}}{2}}$.

7.1 PYTHAGOREAN TRIPLES AND THE SQUARE OF A 2*2 BISYMMETRIC MATRIX.

For x,y real numbers with x>y, and

$$\mathrm{M} = \begin{bmatrix} x & y \\ y & x \end{bmatrix} \tag{25}$$

The square of this matrix is,

$$\mathrm{M}^2 = \begin{bmatrix} x & y \\ y & x \end{bmatrix}^2 = \begin{bmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{bmatrix} . \tag{26}$$

Notice that M^2 has values that are the hypotenuse and altitude of a right triangle whose base is $x^2 - y^2$. As a result, if x and y are integers, then $\{x^2 + y^2, 2xy, x^2 - y^2\}$ is a Pythagorean triple, i.e., three integer sides of a right triangle.

Compare this with the complex number x + iy and its square, $(x+iy)^2 = x^2 - y^2 + 2ixy$. Here the argument of x+iy is doubled while its modulus is squared, i.e., if $\tan\theta = y/x$ then $\tan 2\theta = \dfrac{2xy}{x^2 - y^2}$ while the modulus squares from $\sqrt{x^2 + y^2}$ to $x^2 + y^2$ as shown in Figure 3. The hypotenuse of this triangle is $x^2 + y^2$ so that it is identical with the triangle in Figure 3.

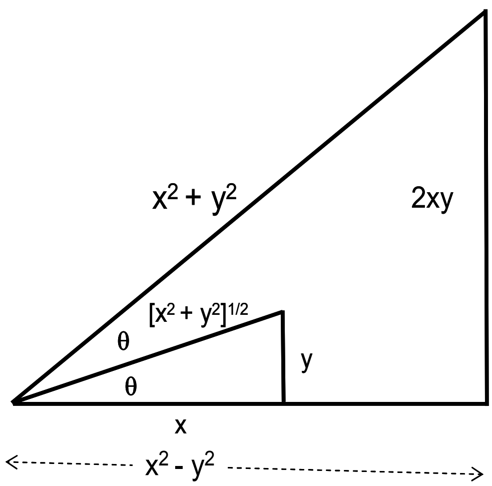

Figure 9. Pythagorean triples

We now identify $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ with the ordered pair (x,y) or equivalently with the complex number x + iy so that $\tan\theta = y/x$ =c. It can be shown that the ordered pair, (c,1), corresponds to a triangle with the radius r of the inscribed circle, and area A given by,

$$r = c-1, \quad A = (c-1)(c)(c+1)$$

It follows that the radius r of the inscribed circle and the area of triangle (a,b) is,

$$r = b^2(a/b - 1) = ab - b^2 \text{ and}$$
$$A = b^4(a/b-1)(a/b)(a/b+1) = ab(a^2 - b^2) \tag{27a,b}$$

It also follows from Equations 27 that,

$$r = \frac{area}{semiperimeter}$$

where this equation holds for triangles that are not right triangles.

7.2 THE SQUARE ROOT OF A 2*2 BISYMMETRIC MATRIX

It follows from Equation 26 that,

$$\begin{bmatrix} x^2+y^2 & 2xy \\ 2xy & x^2+y^2 \end{bmatrix}^{1/2} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

We now pose the problem to find x and y such that,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{1/2} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}. \qquad (28)$$

where a is the hypotenuse and b is the altitude of a right triangle with vertex angle θ whose base is $\sqrt{a^2-b^2}$, $\tan\theta = b/\sqrt{a^2-b^2}$ and $\tan\theta/2 = y/x$. As a result, using standard trigonometric identities,

$$\tan\theta/2 = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad \text{where} \quad \cos\theta = \frac{\sqrt{a^2-b^2}}{a}$$

After some algebra,

$$\tan\theta/2 = \frac{a-\sqrt{a^2-b^2}}{b}$$

which implies that,

$$x = kb \text{ and } y = k(a-\sqrt{a^2-b^2}\ ). \qquad (29)$$

But, since the hypotenuse of the right triangle with vertex $\theta/2$ is $\sqrt{a}$,

$$\sqrt{x^2+y^2} = \sqrt{a} \qquad (30)$$

Replacing Equations 29 into 30 and solving for k it follows after some algebra that,

$$x = \frac{b}{2y} \quad \text{and} \quad y = \sqrt{\frac{a-\sqrt{a^2-b^2}}{2}}$$

which agrees with Equations 8a and b.

7.3 PYTHAGOREAN TRIPLES AND POWERS OF 2X2 BISYMMETRIC MATRICES

For a,b natural numbers with a>b, even powers of an arbitrary 2x2 bisymmetric matrix,

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{2n}$$

result in a sequence of pairs of whole numbers that are hypotenuse and side of Pythagorean triples for all values of n. The third side will be powers of a^2-b^2. If $\sqrt{a^2-b^2} = c$ for c a natural number, i.e., if {a,b,c} is a Pythagorean triple, then all powers of the bisymmetric matrix results in hypotenuse and side of Pythagorean triples with the third side being powers of c. It should be noted that the first number of these

Pythagorean triples, a, represents the length of the hypotenuse unlike the first number of (a,b) which was the length of a side.

Example 1: (3,2)

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^2 = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$$ Therefore the Pythagorean triple is {13,12,5}

$r = 3 \times 2 - 2^2 = 2$ and $A = (3 \times 2)(3^2 - 2^2) = 30$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^4 = \begin{bmatrix} 313 & 312 \\ 312 & 313 \end{bmatrix}$$ with {313,312,25}

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^6 = \begin{bmatrix} 313 & 312 \\ 312 & 313 \end{bmatrix} \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix} = \begin{bmatrix} 7813 & 7812 \\ 7812 & 7813 \end{bmatrix}$$ with {7813,7812,125}

Example 2: a=5, b=4 where {5,4,3} is a triple.

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^2 = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}$$ with {41,40,9}

where $r = 5 \times 4 - 4^2 = 5$ and $A = (5 \times 4)(5^2 - 4^2) = 180$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}^3 = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 365 & 364 \\ 364 & 365 \end{bmatrix}$$ with {365,364,27}

8. CONCLUSION.

Petoukhov's genomic matrices have led to new genomic algebras, generalization of the golden mean, generalized Fibonacci sequences, generalized Nicomachus Triangles, and to an algorithm for generating Pythagorean triples. Pascal's Triangle plays an important role. Although Petoukhov's matrices reproduce the sequences of musical fifths found in the rows of the Nicomachus Triangle, there is no obvious connection between the genomic matrices and the musical scale.

Why it is important to study connections of the genetic code with various algebras? In the beginning of the 19-th century the following viewpoint existed in science: Euclidean geometry was the sole geometry and the arithmetic of real numbers was the sole arithmetic with which to describe the natural world. But this viewpoint was called into question by the discovery of non-Euclidean geometries and the quaternions of Hamilton; this algebra of quaternions presented science with a new multidimensional arithmetic for the study of natural systems [Kline, 1980]. Now science understood that different natural systems can possess their own geometries and algebras. The example of Hamilton, who spent 10 years in his attempts to describe the geometry of 3-dimensional space by means of 3-dimensional algebras without success, presents a cautionary tale. Hamilton's experience shows the importance of developing an appropriate system of algebra with which to describe a natural system. One can add that geometrical and physical-geometrical properties of various natural systems (including laws of conservation, theories of oscillations and waves, theories of potentials and fields, etc.) can depend on the type of algebras which are appropriate to them.

One more important question is the following. Why the genetic code and its degeneracy are organized in such manner that the genetic matrix S_3 (Fig. 1) has the natural algorithmic connection with the matrix Y_8 (Fig. 3) which presents the 8-dimensional algebra? One of possible reasons of such situation is related with a fact that sets of DNA and RNA are the sets which participate in biological reproductions. They are reproduced in a huge number of cells and in chains of generations of organisms. Algebraic operations of addition and multiplication can be useful for such biological sets and for interactions among parts of them. By the way, the matrix Y_8 (Fig. 3) has an interesting property: if modules of all components are equal to 1 ($|x_0|=|x_1|=\ldots=|x_7|=1$) then $Y_8^2=4*Y_8$. In other words, in this case the tetra-reproduction of the genetic matrix Y_8 takes place. It resembles a biological phenomenon of tetra-reproductions of gametes (sex cells) which bear heredity information.

The described 8-dimensional algebras can play also a role of manager layers which determine many features of genetic coding (by analogy with computer technology). Or they can lead to algebras of logical operators of genetic systems. These questions should be investigated by means of using of symmetrological and other methods in the nearest future.

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